HW 3 (Friday)

1. Suppose $h(t) = \sum_{\text{finite}} h_k e^{2\pi i k \cdot t}$, $t \in \mathbb{T}^d \ (= [0,1]^d)$ trigonometric polynomial (of d variables).

Suppose that $h = 0$ on a set of positive Lebesgue measure. Show that $h = 0$ everywhere.

2. Prove a quantitative version of Benedicks' theorem (Amrein-Berthier): $m(A), m(B) < \infty$

$$\Rightarrow \quad \|f\|_2^2 \leq C(A,B)\left(\int_{\mathbb{R}^d \setminus A} |f|^2 + \int_{\mathbb{R}^d \setminus B} |\hat{f}|^2\right), \quad f \in L^2(\mathbb{R}^d)$$

Hint: • It suffices to show that $\|f\|_2^2 \leq C_1(A,B) \int_{\mathbb{R}^d \setminus A} |f|^2$, provided that $\operatorname{spt}(\hat{f}) \subset B$.

• Suppose $\exists (f_n) \subset L^2(\mathbb{R}^d)$, $\|f_n\|_{L^2} = 1$, $\operatorname{spt}(\hat{f}_n) \subset B$, while $\int_{\mathbb{R}^d \setminus A} |f_n|^2 \to 0$. Wlog, $(\hat{f}_n)$ is weakly convergent in L^2, denote by $\hat{f}$ its weak limit. Show that $f_n \to f$ pointwise, an then that $f_n 1_A \to f$ in L^2. Check that $\|f\| = 1$, $\operatorname{spt}(f) \subset A$, $\operatorname{spt}(\hat{f}) \subset B$, which contradicts Benedicks' theorem.

<u>def</u> A sequence $\Lambda = (\lambda_j)_{j\in\mathbb{Z}}$, $\dots < \lambda_{j-1} < \lambda_j < \lambda_{j+1} < \dots$ is separated if there exists $c>0$ such that

$$\lambda_{j+1} - \lambda_j \geq \frac{c}{1+\min(|\lambda_j|, |\lambda_{j+1}|)}, \quad j\in\mathbb{Z}.$$

3. Suppose (Λ, Γ) is a separated uniformly super-critical pair. Then there exist $A, a>0$ s.t.

$$a\|f\|^2_{\mathcal{H}} \leq \sum_{\lambda\in\Lambda} (1+|\lambda|)|f(\lambda)|^2 + \sum_{\gamma\in\Gamma} (1+|\gamma|)|\hat{f}(\gamma)|^2 \leq A\|f\|^2_{\mathcal{H}},$$

for every $f\in\mathcal{H}$.

<u>Hint</u>: to show the left inequality use a "stable version" of the Wirtinger inequality. The right inequality follows from an elementary bound: for $a<b$,

$$|f(a)|^2 \leq C\left[\frac{1}{b-a}\int_a^b |f|^2 + (b-a)\int_a^b |f'|^2\right].$$

def: A discrete set $\Gamma \subset \mathbb{R}$ is ℓ-dense if $\mathbb{R} \setminus \Gamma$ contains no interval of length greater than ℓ.

4.

(a) Let $t > 0$ and let Γ be a $(2t)^{-\frac{1}{2}}$-dense discrete subset of $\mathbb{R}$. Then, for any $\alpha \geq 1$,

$$f \in \mathcal{H},\ f\big|_{\Gamma} = 0 \Rightarrow t^{2\alpha} \int_{\mathbb{R}} |f|^2 \leq \int_{\mathbb{R}} |\xi|^{2\alpha} |\hat{f}(\xi)|^2 .$$

(b) Let Γ be a π-dense discrete subset of $\mathbb{R}$. Then, for any $n \in \mathbb{N}$,

$$f \in S,\ f\big|_{\Gamma} = 0 \Rightarrow \int_{\mathbb{R}} |f|^2 \leq \int_{\mathbb{R}} |f^{(n)}|^2 .$$

Багато успіху !