Cluster Algebras

ICMU

June 24-28, 2024

Khrystyna Serhiyenko

Course Overview

- Lecture 1: Defn and basic properties of cluster Algebras
- Lecture 2: Cluster algebras from surfaces
- Lecture 3: cluster algebras associated to Gassmannians $Gr(k,n)$

Overview

* Grassmannians
* Plabic graphs
* Type A

Cluster Algebras

* Q quiver on vertices $1, \ldots, n$

* $\mathcal{A}_Q$ - cluster algebra $\subset \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ generated by cluster variables that are computed recursively via mutations μ starting from the initial seed $((x_1, \ldots, x_n), Q)$

* designate a subset of vertices of Q mutable and the remaining frozen, then $\mathcal{A}_Q$ is defined as above except mutations at frozen vertices are forbidden

ex/

$$Q = \begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] & & \boxed{3} \arrow[ll] \end{tikzcd}$$

frozen

"$\mathcal{A}_Q$ has the same combinatorics as $\mathcal{A}_{1 \to 2}$"

Grassmannians

* Grassmannian $Gr(k,n)$ is set of k-dim linear subspaces of $\mathbb{C}^n$

* elements of $Gr(k,n)$ can be represented as $k \times n$ matrices $A = k\{\underbrace{[\quad]}_{n}$ of full rank

  row span of $A \rightsquigarrow$ element of $Gr(k,n)$

* $Gr(k,n) = GL_k \backslash Mat^*_{k \times n}(\mathbb{C})$

  modulo row operations

  full rank $k \times n$ matrices

Grassmannians

* $I \in \binom{[n]}{k}$ k-elt subset of $[n] = \{1, .., n\}$
* $p_I(A)$ Plücker coordinate i.e. maximal minor of A on columns indexed by I
* Plücker embedding : $Gr(k,n) \hookrightarrow \mathbb{CP}^{\binom{n}{k}-1}$

$$A \longmapsto (p_I(A))_{I \in \binom{[n]}{k}}$$

Gr(k,n) can be identified with its image, and is a projective variety

* $\mathbb{C}[Gr(k,n)]$ coordinate ring of the Grassmannian

i.e. elements of $\mathbb{C}[Gr(k,n)]$ are functions on $Gr(k,n)$

Grassmannians

Th $\mathbb{C}[Gr(k,n)] = \langle p_I \mid I \in \binom{[n]}{k} \rangle \Big/ \text{Plücker relations}$

ex $\mathbb{C}[Gr(2,4)] = \langle p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34} \rangle \Big/ p_{13}p_{24} = p_{12}p_{34} + p_{14}p_{23}$

$$A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix} \in Gr(k,n)$$

$$p_{13}(A)\,p_{24}(A) \overset{?}{=} p_{12}(A)\,p_{34}(A) + p_{14}(A)\,p_{23}(A)$$

$$\begin{vmatrix} 1 & a \\ 0 & c \end{vmatrix} \cdot \begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix} \overset{?}{=} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix} \cdot \begin{vmatrix} 0 & a \\ 1 & c \end{vmatrix}$$

$$c \cdot (-b) \overset{\checkmark}{=} 1 \cdot (ad - bc) + d \cdot (-a)$$

Grassmannians

Th [Scott] $\mathbb{C}[Gr(k,n)]$ is a cluster algebra

$$\{\text{Plücker coordinates}\} \subset \{\text{cluster variables}\}$$

exchange relation

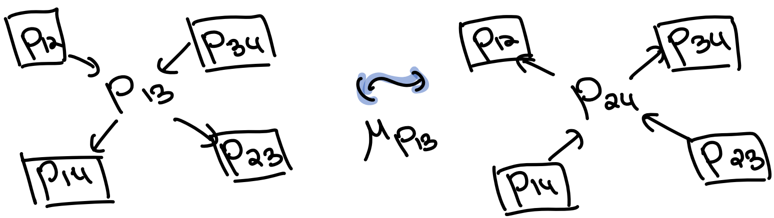
ex

$$\mathbb{C}[Gr(2,4)] = \langle p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34} \rangle \big/ p_{13}p_{24} = p_{12}p_{34} + p_{14}p_{23}$$

$$\mathbb{C}[Gr(2,4)] \subset \mathbb{C}(p_{12}, p_{14}, p_{23}, p_{34}, p_{13})$$

ambient field

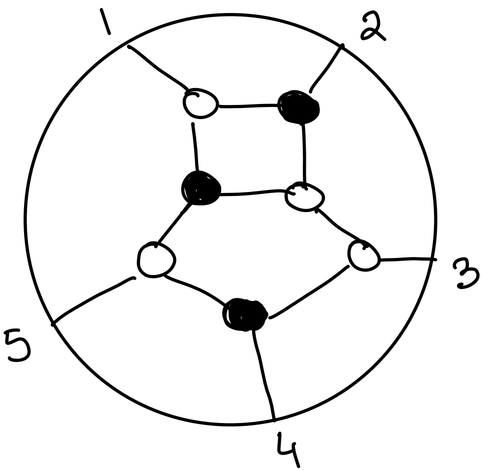
* two seeds
* $p_{12}, p_{34}, p_{14}, p_{23}$ frozen cluster variables
* p_{13}, p_{24} mutable cluster variables

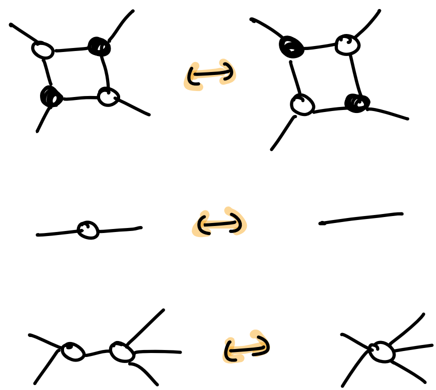


Plabic Graphs

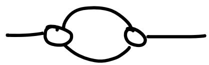
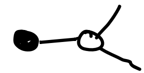
* G - plabic (planar bi-colored) connected graph drawn inside a disk

local moves on G

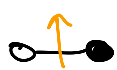




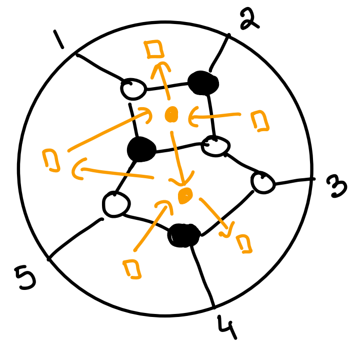
Plabic Graphs

* G is reduced if it cannot be transformed via local moves to a graph containining a digon  or internal leaf 

* Q(G) – quiver associated to G

G	Q(G)
internal faces	mutable vertices 
boundary faces	frozen vertices 
	arrows 

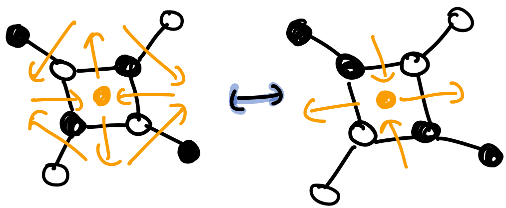
ex/



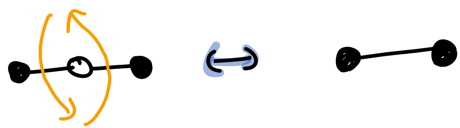
Plabic Graphs

* square move corresponds to quiver mutation while other local moves preserve the quiver

ex



mutation

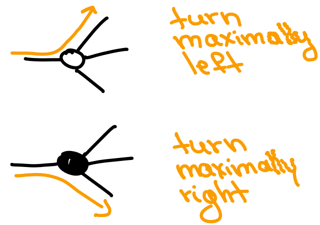


no change

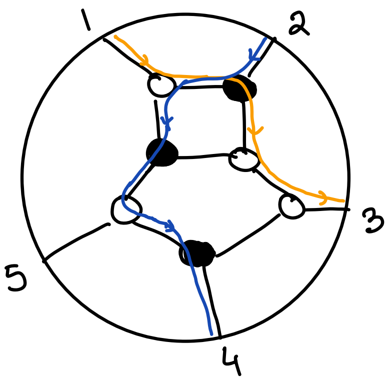
no change

Plabic Graphs

reduced plabic graph G $\rightsquigarrow$ trip permutation π_G



ex



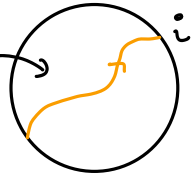
$$\pi_G = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{matrix}$$

Th: reduced plabic graphs are move equivalent iff they have the same trip permutation

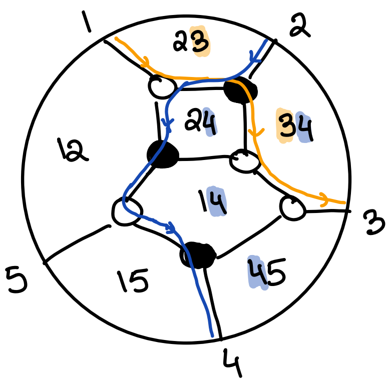
Plabic Graphs

* label faces of G by subsets I s.t. $i \in I$ if the face is to the left of the trip ending in i

faces here contain label i



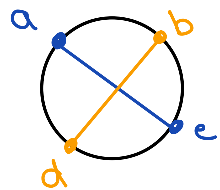
ex



Th face labels have the same cardinality

Plabic Graphs

* $I, J \in \binom{[n]}{k}$ are weakly separated if there do not exist $a, c \in I \setminus J$ and $b, d \in J \setminus I$ s.t. a, b, c, d is cyclically ordered



not weakly separated

Th ① Face labels of a reduced plabic graph are pairwize weakly separated.

② Any pairwize weakly separated collection can be realized as face labels of a reduced plabic graph.

Plabic Graphs

Th [Scott] $\mathbb{C}[Gr(k,n)]$ is a cluster algebra

$$\left\{\begin{matrix}\text{Plücker}\\ \text{coordinates}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{cluster}\\ \text{variables}\end{matrix}\right\}$$

key: understand seeds of the cluster algebra consisting of Plücker coordinates in terms of plabic graphs

* G is of type (k,n) if $\pi_G \in S_n$ and $\pi_G(i) = i+k$

Plabic Graphs

Th Cluster structure on $\mathbb{C}[Gr(k,n)]$

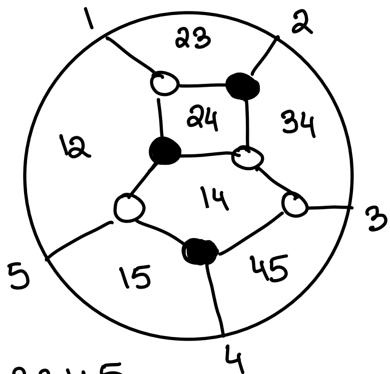
$\{$reduced plabic graphs G of type $(k,n)\}/\sim$ $\longleftrightarrow$ $\{$seeds consisting of Plücker coordinates $(\tilde{x}_G, Q(G))\}$

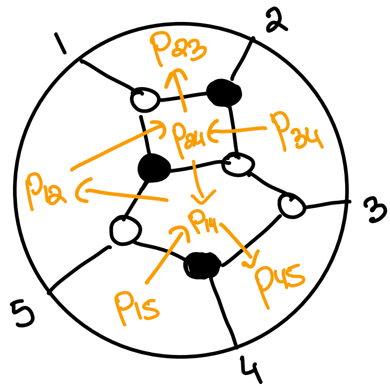
$\{$reduced plabic graphs G of type $(k,n)\}/\sim$ $\longleftrightarrow$ $\{$max. pairwise weakly separated collections in $\binom{[n]}{k}\}$

$\{$seeds consisting of Plücker coordinates $(\tilde{x}_G, Q(G))\}$ $\longleftrightarrow$ $\{$max. pairwise weakly separated collections in $\binom{[n]}{k}\}$

Plabic Graphs

ex





$\pi_G = \begin{matrix} 1\,2\,3\,4\,5 \\ 3\,4\,5\,1\,2 \end{matrix}$

type $(2,5)$

$\left\{ \begin{matrix} 12, 23, 34, 45, 15, \\ 24, 14 \end{matrix} \right\}$

Type A

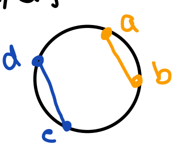
* A Q is of type A_n if Q is mutation equivalent to

  $$1 \to 2 \to \cdots \to n$$

* $\mathbb{C}[Gr(2,n)]$ is of type A_{n-3}

  $$\{\text{Plücker coordinates}\} \longleftrightarrow \{\text{cluster variables}\}$$

* $I, J \in \binom{[n]}{2}$ are weakly separated iff $I=\{a,b\}$, $J=\{c,d\}$ s.t. a, b, c, d are cyclically ordered i.e.

  I and J are noncrossing diagonals in an n-gon

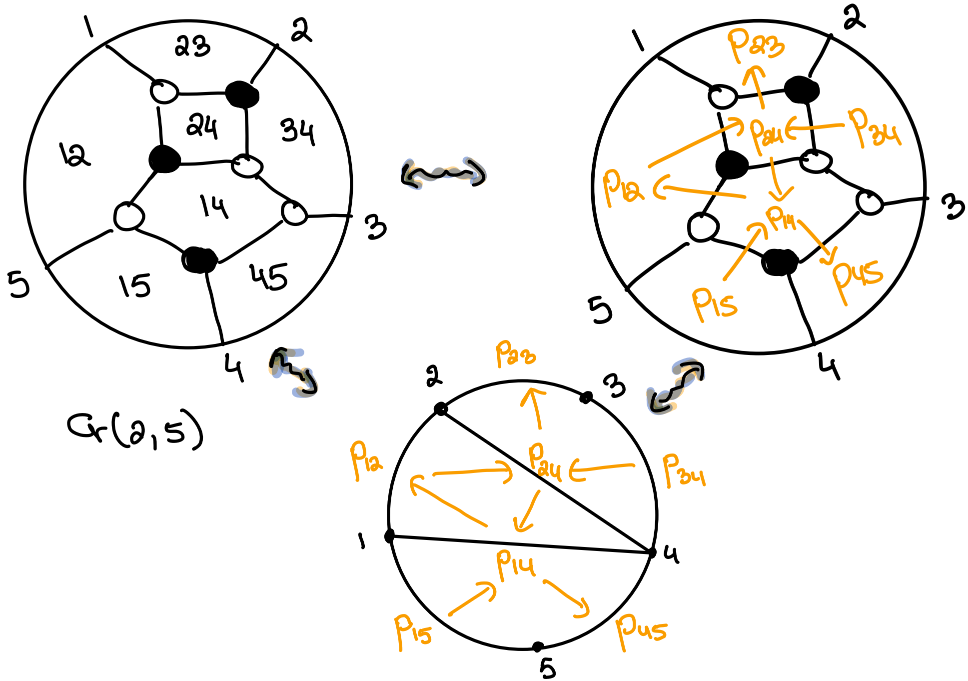
Type A

Th Cluster structure on $\mathbb{C}[Gr(2,n)]$

$$
\begin{tikzcd}
\left\{\begin{array}{l}\text{reduced plabic}\\ \text{graphs } G\\ \text{of type } (2,n)\end{array}\right\}\Big/\sim \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \left\{\begin{array}{l}\text{seeds}\\ (x_G, Q(G))\end{array}\right\} \arrow[dl, leftrightarrow] \\
& \left\{\begin{array}{l}\text{max. collections}\\ \text{of non-crossing}\\ \text{diagonals in } n\text{-gon}\\ \text{i.e. triangulations}\end{array}\right\} &
\end{tikzcd}
$$

Type A

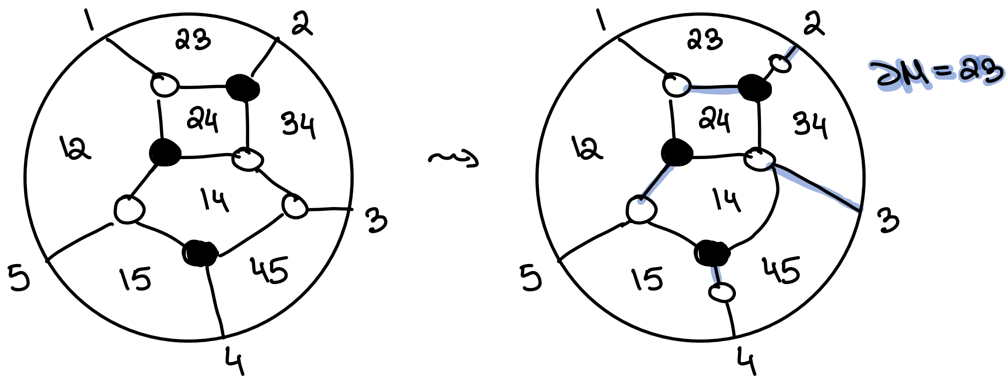
ex



$Gr(2,5)$

Type A

* transform G s.t. boundary edges are adjacent to white vertices and G is bipartite



* M is a matching of G if all interior vertices are covered by edges of M
* ∂M - collection of boundary edges of M

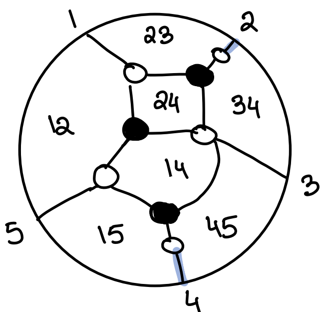
Type A

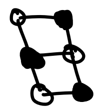
* can compute cluster variables by looking at matchings of G with weights on the edges of G starting with the initial seed given by G

* $P_{ab} = \sum_{\substack{M \\ \partial M = \{a-1,\, b-1\}}} w(M)$ where $w(M)$ is a weight of M

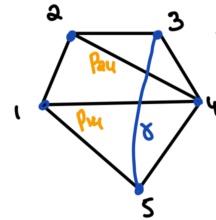
* $P_{ab} = \frac{1}{\text{cross}(\gamma)} \sum_{P \in \text{Match}(\gamma)} x(P)$ expansion formula

ex





γ associated to P_{35}



$\gamma \rightsquigarrow P_{35}$

$P_{35} = \sum_{\substack{M \\ \partial M = 24}} w(M)$

Thank you !