ICMU Mini course:
Introduction to Cluster algebras

Homework 3

1. Define totally positive matrix to be a matrix in which all the minors are positive: that is, the determinant of every square submatrix is a positive number.

   (a) What are the minimal subsets of minors which are enough to test for positivity to ascertain that 2×2 matrix is totally positive? What is the number of such subsets? What is the total number of minors which appear in all such subsets.

   (b) Same question as in part (a) for 2×3 matrix.

2. Let $1 \leq m \leq n$ be integers. The set of $n \times m$ matrices can be embedded inside in $\mathrm{Gr}(m, m+n)$ via:

$$\{\text{all } m \times n \text{ matrices}\} \hookrightarrow \mathrm{Gr}(m, m+n) \quad \text{where} \quad A \mapsto \overline{A}$$

$$\overline{A} := \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & (-1)^{m-1}a_{m1} & (-1)^{m-1}a_{m2} & \dots & (-1)^{m-1}a_{mn} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & a_{31} & a_{32} & \dots & a_{3n} \\ 0 & 0 & \dots & 0 & 1 & 0 & -a_{21} & -a_{22} & \dots & -a_{2n} \\ 0 & 0 & \dots & 0 & 0 & 1 & a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}.$$

   (a) The embedding defined above induces a one-to-one correspondence between all minors $A_{P,Q}$ of A, $P \subseteq [m], Q \subseteq [n]$, $|P| = |Q|$, and all maximal minors $\overline{A}_{[m],I}$ of $\overline{A}$, $I \subseteq [m+n]$, $|I| = m$, such that

$$\det A_{P,Q} = \det \overline{A}_{[m],I},$$

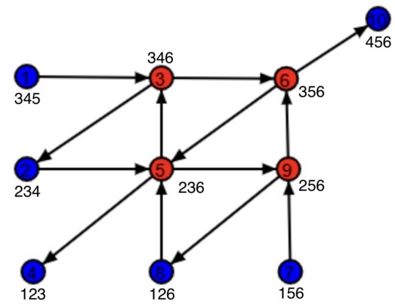
   where we follow convention that determinant of an empty minor $\det_{\emptyset,\emptyset}(A) = 1$. Describe explicitly subset I in terms of sets P and Q given by the above correspondence.

   (b) Consider a set $\mathbf{T}$ of minors $\det(A_{[a_1,b_1],[a_2,b_2]})$ indexed by pairs of intervals, at least one of which contains 1. Show that $\mathbf{T}$ correspond to a maximal weakly-separated set of Plücker coordinates of $\overline{A}$.

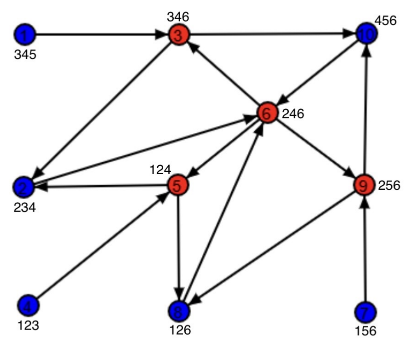
   (c) Prove that to ascertain total positivity of A it suffices to verify the positivity of the minors in $\mathbf{T}$.

   (d) Compare the size of the set $\mathbf{T}$ with the total number of minors in matrix A.

3. Below you can see quiver of initial seed of cluster algebra of $\mathrm{Gr}(3,6)$, where mutable variables are red and frozen variables are blue. Notice that Plücker coordinates in the seed are weakly-separated as it should be in any cluster consisting of Plücker coordinates.



(a) Show that the quiver above Q_1 can be transformed to the quiver below Q_2 by a sequence of mutations.



Notice that the mutable part of Q_2 above is an orientation of the Dynkin diagram D_4. Cluster algebra of $\mathrm{Gr}(3,6)$ contains two variables which are not Plücker coordinates:

$$x_1 = [134][256] - [234][156] \qquad x_2 = [245][136] - [126][345].$$

(b) Verify that the exchange relation of the mutation at vertex 246 of the Q_2 involves cluster variable x_1.

(c) Prove that for any 3×3 totally positive matrix the following inequalities hold:

i. $\det A_{13,23} \det A_{2,1} - \det A_{12,23} \det A_{3,1} \geq 0$

ii. $\det A_{13,12} \det A_{2,3} - \det A_{23,12} \det A_{1,3} \geq 0$

Similar observations can be done for $\mathrm{Gr}(3,7)$ and $\mathrm{Gr}(3,8)$. Their cluster algebras are of finite type and correspond to E_6 and E_8. Beyond cases $\mathrm{Gr}(2,n), \mathrm{Gr}(3,6), \mathrm{Gr}(3,7), \mathrm{Gr}(3,8)$ cluster algebras of Grassmannians have infinitly many cluster variables.

In general, the initial standard cluster of $\mathrm{Gr}(m,n)$ looks similar to Q_1. See J. Scott Grassmannians and Cluster Algebras for more details.

Observe that the minimal subsets of minor which are enough to test for positivity of matrix A in the very first exercise correspond to seeds of cluster algebras of $M_{2\times 2}$ and $M_{2\times 3}$. Such cluster algebras have same combinatorics as those for $\mathrm{Gr}(2,4)$ and $\mathrm{Gr}(2,5)$. The standard initial quiver is almost the same as for Grassmannian cases except the vertices with frozen variables 34 and 456 are removed.