HW-1 (Monday)

We use the Fourier transform normalized by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\, dx,$$

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi.$$

1. $f \in L^2(\mathbb{R})$, $|f(x)| \lesssim e^{-a|x|}$

$\Rightarrow \hat{f}$ is analytic in $\{|\mathrm{Im}(\xi)| < \frac{a}{2\pi}\}$

2.

(a) Prove the Heisenberg inequality

$$\|xf\|_2 \, \|\xi \hat{f}\|_2 \geq \frac{d}{4\pi} \|f\|_2^2, \qquad (*)$$

assuming that $f \in S(\mathbb{R}^d)$.

<u>Hint</u>: Start with $\sum_{j=1}^{d} x_j \partial_j |f|^2 = 2\,\mathrm{Re}\,(\nabla f \cdot x\bar{f})$.
The rest is the same as in the proof of 1D case given in the class.

(b) Describe the functions for which the equality sign attains in $(*)$.

<u>def</u>: Sobolev space:

$$H_1 := \{ f \in L^2 : \ \xi \hat{f} \in L^2 \}, \quad \|f\|_{H_1}^2 = \int_{\mathbb{R}} (1+\xi^2) |\hat{f}|^2.$$

3.

(a) Suppose $f \in H_1$. Then $\|\hat{f}\|_{L^1} \leq C \|f\|_{H_1}$. Conclude that functions in H_1 are continuous and tend to infinite as $x \to \pm\infty$.

(b) Let $\mathcal{H} := \{ f : \ f, \hat{f} \in H_1 \}$, $\|f\|_{\mathcal{H}}^2 = \|f\|_{H_1}^2 + \|\hat{f}\|_{H_1}^2$. Show that the pt evaluations $x \mapsto f(x)$ and $\xi \mapsto \hat{f}(\xi)$ are bdd functionals in $\mathcal{H}$, and that the Schwartz space $\mathcal{S}$ is dense in $\mathcal{H}$.

4. Prove the Heisenberg inequality

$$\|x f\|_2 \cdot \|\xi \hat{f}\|_2 \geq \frac{1}{4\pi} \|f\|_2^2, \qquad \text{for } f \in L^2(\mathbb{R}).$$

<u>Hint</u>: Put $f' := [2\pi i \xi \hat{f}]^{\vee}$, $f' \in L^2$ ($'$ is a symbol here). You need to justify integration by part, i.e., that

$\int_{\mathbb{R}} x(f'\bar{f} + \bar{f}'f) = -\|f\|_2^2$; the rest is similar to the case when $f \in S'$ discussed in the class. To justify this, approximate f by a sequence $(f_n) \subset S$ so that

$$\int_{\mathbb{R}} (1 + 4\pi\xi^2)\,|\hat{f}_n - \hat{f}\,|^2 \to 0 \text{ for } n \to \infty.$$

Бажато успіху!