24–28 June, 2024

ICMU Mini Course:
Introduction to Cluster Algebras

Homework 1

1. Show that the mutation operation μ_k is an involution on the seeds, that is $\mu_k \circ \mu_k = 1$.

2. Compute the exchange graph for the cluster algebra $\mathcal{A}_Q$ where the quiver Q is $1 \to 2$.

3. Let Q^{op} denote the quiver obtained from a quiver Q by reversing the orientation of all of its arrows. How are the two cluster algebras $\mathcal{A}_Q$ and $\mathcal{A}_{Q^{\mathrm{op}}}$ related? Prove your claim.

4. In each case provide an example of a quiver Q that satisfies the given condition. Briefly justify your answers.
   (a) Q has oriented cycles but the cluster algebra $\mathcal{A}_Q$ is acyclic.
   (b) Q is such that every quiver that is mutation equivalent to Q is isomorphic to Q.
   (c) $\mathcal{A}_Q$ is of finite mutation type but not of finite type.
   (d) $\mathcal{A}_Q$ is of acyclic type but not of finite mutation type.

5. Consider a recurrence defined by $z_{k-1}z_{k+1} = z_k^2 + 1$ with the initial condition $z_1 = z_2 = 1$. Show that the terms of the recurrence are positive integers.

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Homework 2

1. Prove that the number of clusters in a cluster algebra of type A is given by the Catalan numbers C_n, i.e. $C_0 = 1, C_1 = 1, C_2 = 2$ and they satisfy the recurrence $C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}$. What is the number of cluster variables in a cluster algebra of type A?

2. Let Q_T be a quiver coming from a triangulation of a surface. What is the largest number of arrows starting/ending at a vertex of Q_T? Use this to show that Q_T is of finite mutation type.

3. Prove that cluster algebras coming from triangulations of annuli are of acyclic type.

4. Let $\mathcal{G}_d$ be a snake graph on $d \geq 1$ tiles. For each of the following families of snake graphs find and prove a formula for the number of matchings of $\mathcal{G}_d$.
   (a) Every three adjacent tiles of $\mathcal{G}_d$ form a straight-piece configuration.
   (b) Every three adjacent tiles of $\mathcal{G}_d$ form a zig-zag configuration.

5. Consider a triangulation T of the surface (S, M). Let γ be the red arc as in the picture. Find the cluster variable x_γ corresponding to the arc γ in the cluster algebra $\mathcal{A}_{Q_T}$.