ICMU
INTERNATIONAL CENTRE FOR MATHEMATICS IN UKRAINE

ICMU Winter School

RANDOM MATRICES, RANDOM ANALYTIC FUNCTIONS, AND NON-LINEAR PDEs

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Abstracts of minicourses

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Statistics of eigenvectors of non-Hermitian random matrices

A (square) matrix X is called non-normal if it does not commute with its Hermitian adjoint: $XX^* \neq X^*X$. Generically, non-Hermitian random matrices are non-normal. To each eigenvalue of a non-normal matrix λ_i, real or complex, correspond two eigenvectors: *left* $\mathbf{l}_i$ and *right* $\mathbf{r}_i$. The corresponding eigenproblems are

$$X\mathbf{r}_i = \lambda_i \mathbf{r}_i \text{ and } X^*\mathbf{l}_i = \overline{\lambda}_i \mathbf{l}_i.$$

The two sets can always be chosen *bi*-orthogonal: $(\mathbf{l}_i^* \mathbf{r}_j) = \delta_{ij}$.

Consider now an (additively) perturbed matrix $X' = X + \epsilon P$, with $\epsilon > 0$ controlling the magnitude of the perturbation P. To the leading order in ϵ the eigenvalues are shifted by

$$|\lambda_i(\epsilon) - \lambda_i(0)| = \epsilon |\mathbf{l}_i^* P \mathbf{r}_i| \leq \epsilon ||P||_2 \sqrt{(\mathbf{l}_i^* \mathbf{l}_i)(\mathbf{r}_i^* \mathbf{r}_i)} \quad ,$$

showing that the sensitivity of eigenvalues is mainly controlled by the *eigenvalue condition numbers*:

$$\kappa_i = \sqrt{(\mathbf{l}_i^* \mathbf{l}_i)(\mathbf{r}_i^* \mathbf{r}_i)} \geq 1,$$

with $\kappa = 1$ only when X is normal. Thus, non-normal matrices are much more sensitive to perturbations of the matrix entries than their normal counterparts. Moreover, for big random matrices this sensitivity may become parametrically strong. Further, the condition numbers in random case turn out to be broadly distributed.

The goal of the course will be to provide a detailed discussion of statistics of condition numbers, as well as some related quantities, for the large $N \times N$ random matrices from the simplest complex Ginibre ensemble, where entries are i.i.d. complex normals: $\Re X_{j,k} \sim \Im X_{j,k} \sim N^{-1/2}\mathcal{N}(0, 1/2)$. The technique employed will be a combination of the incomplete Schur decomposition and a variant of the so-called supersymmetry method which uses integration over anticommuting variables. The discussion of these techniques, especially in the limit of large N, will be presented at an (in)formal level, with the emphasis on underlying ideas rather than on the rigour of the presentation.

Leonid Pastur

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Large sample covariance matrices and their application to deep learning networks and quantum gravity

Large empirical covariance matrices are the object of active research in a number of areas of modern mathematics and related fields. They originated in the 1930s in classical mathematical statistics, whose goal was to statistically predict the values of a finite number p of unknown parameters from the results of a large number of $n \to \infty$ independent measurements (samples), that is, to operate with covariance matrices of size p, in particular, with their eigenvalues as $n \to \infty$. However, with the advent of the era of big data, when p can be as large as n, it became clear that classical methods can lead to erroneous predictions, and so the problem of studying large covariance matrices, in particular those where $p \to \infty$, $n \to \infty, p/n \to c \in (0, \infty)$ arose. In the 1950s-1960s, large random matrices of a somewhat different type appeared in theoretical physics. This gave rise to the theory of random matrices, which has been actively developed ever since. The proposed course will present a simple and general method to obtain the eigenvalue distributions of empirical covariance matrices in the above double-scaling limit. The method is based on elementary facts of linear algebra and probability theory. A brief overview of applications of these and other large random matrices in theoretical physics, quantum information theory, and neural networks will also be given.

Mariya Shcherbina

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Supersymmetric approach in modern mathematics

Since the end of the last century, methods of supersymmetry, based on the simultaneous use of commuting and anti-commuting variables, have been widely used in modern theoretical physics. The methods proved to be very efficient in analysis of disordered (random) systems, allowing researchers to obtain many interesting results.

However, an active application of the supersymmetric methods in mathematics is still in its early stages, although the number of examples of such applications grows every year. The approach allows researchers to solve many interesting long-standing problems by using the number of algebraic tricks that are very similar to the standard linear algebra and Gaussian integration methods, although they involve the operations with anti-commuting variables. As it was written in the preface to Efetov's book on supersymmetry: "All supersymmetric theories are based on the use of anti-commuting variables introduced by Grassmann in the last century. At the first glance these objects look very artificial and seem to have no relation to the real world. There is a certain threshold to start using Grassmann anti-commuting variables in the calculations because one expects that they should be very unusual. Surprisingly, it is not true and, provided proper definitions are given, one can simply generalize conventional mathematical constructions so that it is possible to treat both commuting and anti-commuting variables on an equal footing".

The course aims to give an introduction into the mathematical foundations of supersymmetric approach. It includes the definition of anti-commuting Grassamann variables, super vectors, super matrices, and operations with them. We introduce integration over Grassmann variables and compute some important integrals. Finally, some applications to random matrix theory will be presented.

Necessary prerequisites:

- basics of linear algebra,
- basics of probability theory.

Dmitry Shepelsky
ILTPE, Kharkiv

Integrable nonlinear partial differential equations and Riemann-Hilbert problems

Riemann-Hilbert (RH) problems are boundary-value problems for (piece-wise) analytic functions in the complex plane. It is a remarkable fact that a vast array of problems in mathematics, mathematical physics, and applied mathematics can be posed as Riemann-Hilbert problems. These include radiation, elasticity, hydrodynamic, diffraction problems, orthogonal polynomials and random matrix theory, nonlinear ordinary and partial differential equations. In applications, the data for a RH problem depend on external parameters, which turns out to be physical variables (space, time, matrix size, etc) in the relevant equations modeling particular processes. Particularly, we are interested in the application of the RH problem approach to a class of nonlinear partial differential equations (PDE) called "integrable equations" or "soliton equations". This class of PDE has been the subject of intensive studies since 1970th, when the so-called Inverse Scattering Transform (IST) method was invented. The RH problem approach can be viewed as a particular realization of the IST method. The representation of a solution to a nonlinear partial differential equation in terms of a solution of the associated RH problem can be viewed as a nonlinear analogue of the contour integral representation for a linear PDE. Particularly, it provides effective means to evaluate accurately the solutions in various asymptotic regimes. The lectures aim at introducing the Inverse Scattering Transform method in the form of a Riemann-Hilbert problem for studying integrable nonlinear differential equations and illustrating the fruitfulness of the method by studying the long-time asymptotics of solutions of such equations. As a prototype model, we use the nonlinear Schrödinger equation, which is a basic model of nonlinear wave propagation in various situations.

Necessary prerequisites:

- basics of the complex analysis;
- basics of the theory of ordinary differential equations.

Mikhail Sodin

Tel Aviv University

Random zero sets

Random polynomials and analytic functions have attractedthe attention of mathematicians for a long time, though the focus of interest has changed over time. Just as the distribution of eigenvalues is central to random matrix theory, the zero sets are central to the study of random analytic functions. In recent decades, the subject has been revived due to numerous new links to mathematical physics, probability theory, complex and harmonic analysis, and complex geometry. In this mini-course, I will try to provide a glimpse into classical and recent results and open questions pertaining to the zero sets of random polynomials and random analytic functions.

Bálint Virág

University of Toronto

Operator limits of random matrices

We will discuss matrix models for beta ensembles, as well as their infinite limits. The limits are random operators; studying these operators will give access to asymptotic questions about the eigenvalue distribution of finite matrices.

The notes for this topic can be found on the webpage
`https://www.math.toronto.edu/~balint/ViragNotes2.pdf`