HW 2 (Wednesday)

1. Deduce the functional equation for Riemann's ζ-function:

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-(1-s)/2}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \quad s\in\mathbb{C}$$

Hint: Consider $g(t):=\sum_{n\geq 1} e^{-\pi n^2 t}$

- Show that $g(t)=-\frac{1}{2}+\frac{1}{2\sqrt{t}}+\frac{1}{\sqrt{t}}\,g\left(\frac{1}{t}\right)$

(the Poisson summation)

- $\mathrm{Re}(s)>\frac{1}{2}$

$$\int_0^\infty g(t)\,t^{s-1}\,dt=\frac{\Gamma(s)}{\pi^s}\zeta(2s)$$

$$\Rightarrow \frac{\Gamma(s/2)}{\pi^{s/2}}\zeta(s)=\frac{1}{s-1}-\frac{1}{s}+\int_1^\infty g(t)\left[t^{s/2}+t^{(1-s)/2}\right]\frac{dt}{t}$$

- the integral on the RHS is an entire f-n
- RHS is invariant w.r.t. $s\mapsto 1-s$.

Remark: This was one of Riemann's original proofs. Note that it uses only that $\zeta(s)$ is defined and analytic only for $\mathrm{Re}(s)>1$ where $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$.

2. $PW_{1/2} := \{f \in L^2(\mathbb{R}) : \operatorname{spt}(\hat{f}) \subset [-\frac{1}{2}, \frac{1}{2}]\}$ - the Paley-Wiener space.

(a) Show that $PW_{1/2}$ is a closed subspace of $L^2(\mathbb{R})$.

(b) $f \in PW_{1/2} \Rightarrow f$ is entire, $|f(z)| \le \|f\|_2 \, e^{\pi |\operatorname{Im} z|}$.

(c) f is entire, $|f(z)| \le C(1+|z|)^{-1} e^{\pi |\operatorname{Im} z|} \Rightarrow f \in PW_{1/2}$

(d) Show that the system of functions $\left(\frac{\sin \pi(x-k)}{\pi(x-k)}\right)_{k \in \mathbb{Z}}$ is an orthonormal basis in $PW_{1/2}$, and that the sampling formula (*) provides a Hilbert spaces isomorphism $\ell^2(\mathbb{Z}) \to PW_{1/2}$.

(e) Put $S(t, z) = \dfrac{\sin \pi(t-\bar{z})}{\pi(t-\bar{z})}$, $t \in \mathbb{R}$, $z \in \mathbb{C}$.
Then, for any $f \in PW_{1/2}$, $f(z) = \langle f, S(\cdot, z) \rangle_{L^2(\mathbb{R})}$, i.e. $S(t, z)$ is the reproducing kernel in $PW_{1/2}$.

<u>Remark</u>: (c) is a weak version of the Paley-Wiener theorem: f is entire, $|f(z)| \le Ce^{\pi |z|}$, $f \in L^2(\mathbb{R}) \Rightarrow f \in PW_{1/2}$.

The general version can be deduced (in several steps) from (c) using the Phragmén-Lindelöf principle.

3. Prove a "polynomial version" of Hardy's thm:

$$|f(x)| \lesssim (1+|x|^n)e^{-\pi x^2}, \quad |\hat{f}(\xi)| \lesssim (1+|\xi|^2)^n e^{-\pi \xi^2}$$

$$\Rightarrow f = Pe^{-\pi x^2}, \quad \hat{f} = Qe^{-\pi x^2}, \quad \deg P, \deg Q \le n.$$

4. Prove a "one-sided version" of Hardy's thm:

Suppose that $|f(x)| \lesssim e^{-ax^2}$, $x \ge 0$, and $|\hat{f}(\xi)| \lesssim e^{-b\xi^2}$ for $\xi \ge 0$. Then $f = 0$ provided that $a \cdot b$ is sufficiently large.

Hint:

- Wlog, $f \in S$ (otherwise, convolve f with a test function $\chi \in C_0^\infty(\mathbb{R})$, then do the same on the Fourier side)
- Set $\tilde{f}(x) = f(-x)$, consider the function $(f \cdot \bar{\tilde{f}}) * (\bar{f} \cdot \tilde{f})$, and estimate its (two-sided) decay.

Бажаю успіху!