Fourier uncertainty and uniqueness

Mini-course, June 2024.

Program:

(1) Heisenberg uncertainty principle

(2) Hardy's theorem

(3) Benedicks' theorem

(4) Discrete Fourier uniqueness pairs.

Preliminaries:

- The Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\xi}\, dx \qquad \left(\widehat{e^{-\pi x^2}} = e^{-\pi\xi^2}\right)$$

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi$$

- Spaces: $L^2 = L^2(\mathbb{R}^d)$, $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ (Schwartz space)

Sobolev space $\mathcal{H}_1$:

$$\|f\|^2_{\mathcal{H}_1} = \int_{\mathbb{R}^d} (1+|\xi|^2)\, |\hat{f}(\xi)|^2\, d\xi$$

$$\simeq \sum_{|\alpha|\leq 1} \|D^\alpha f\|^2_{L^2}$$

$H_1 = \{f : f, \hat{f} \in \mathcal{H}_1\}$ "invariant Sobolev space"

- Example: $\left.\begin{array}{l} f \in L^2(\mathbb{R}^d) \\ f, \hat{f} \text{ have compact spt} \end{array}\right\} \Rightarrow f = 0.$

1. Heisenberg uncertainty principle:

Thm (I): $\|xf\|_2 \cdot \|\xi \hat{f}\|_2 \geq \frac{d}{4\pi} \|f\|_2^2$. Equality attains iff $f(x) = Ce^{-(Ax,x)}$, $A = \mathrm{diag}(a_1, \dots, a_d)$, $a_j > 0$.

• Hilbert space / quantum mechanics approach

$\mathcal{H}$ Hilbert space, A unbdd operator

Example: $f \mapsto x_j \cdot f$, $f \mapsto \frac{1}{2\pi i} \partial_j f$, $1 \leq j \leq d$, $f \in L^2$

$\mathcal{D}(A) \subset \mathcal{H}$ domain of A, dense in $\mathcal{H}$

$\mathcal{D}(A^*) = \{ g \in \mathcal{H} : \exists h \in \mathcal{H} \ \langle Af, g \rangle = \langle f, h \rangle, \ \forall f \in \mathcal{D}(A) \}$

(then, $A^* g \underset{\text{def}}{=} h$)

• A is symmetric (hermitean) if

$\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and $Af = A^* f$, $\forall f \in \mathcal{D}(A)$

$(\Leftrightarrow \langle Af, g \rangle = \langle f, Ag \rangle, \ \forall f, g \in \mathcal{D}(A))$

• A is self-adjoint if it is symmetric and

$\mathcal{D}(A) = \mathcal{D}(A^*)$

- $[A,B] = AB - BA$ (acts on $\mathcal{D}(AB) \cap \mathcal{D}(BA)$)

Thm (II): A, B self-adjoint

$\Rightarrow \|Af\| \cdot \|Bf\| \geq \frac{1}{2} |\langle [A,B]f, f\rangle|$, $f \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$.

The equality sign attains iff $Af = icBf$, $c \in \mathbb{R}$

Pf of (II):

$$\langle [A,B]f, f\rangle = \langle Bf, Af\rangle - \langle Af, Bf\rangle$$

$$= \langle Bf, Af\rangle - \overline{\langle Bf, Af\rangle} = 2\,\mathrm{Im}\,\langle Bf, Af\rangle$$

$$\Rightarrow |\langle [A,B]f, f\rangle| \leq 2 |\langle Bf, Af\rangle|$$

$$\leq 2 \|Bf\| \cdot \|Af\|.$$

Equality sign: $\langle Bf, Af\rangle \in i\mathbb{R}$ & $Af = \lambda Bf$, $\lambda \in \mathbb{C}$

$\Leftrightarrow \lambda = ic$, $c \in \mathbb{R}$. $\square$

Remark: $a, b \in \mathbb{R} \Rightarrow [A-a, B-b] = [A,B]$

$$\Rightarrow \|(A-a)f\| \cdot \|(B-b)f\| \geq \frac{1}{2} |\langle [A,B]f, f\rangle|$$

<u>(II) ⇒ (I):</u> $\mathcal{H} = L^2(\mathbb{R}^d)$

$$A_j f = x_j f, \quad B_j f = \frac{1}{2\pi i} \partial_j f, \quad 1 \le j \le d$$

$$[A_j, B_j] = \frac{1}{2\pi i}(x_j \partial_j - \partial_j x_j) = -\frac{1}{2\pi i} I$$

$$\Rightarrow \frac{1}{2}\|f\|_2^2 \le \|x_j f\|_2 \cdot \|\partial_j f\|_2$$

$$\Rightarrow \frac{d}{2}\|f\|_2^2 \le \sum_{j=1}^{d} \|x_j f\|_2 \, \|\partial_j f\|_2$$

$$\le \left(\sum_{j=1}^{d} \|x_j f\|_2^2 \cdot \sum_{j=1}^{d} \|\partial_j f\|_2^2 \right)^{1/2}$$

$$= \|x f\|_2 \cdot \|2\pi \xi \hat{f}\|_2 .$$

<u>Ex:</u> Check the case of equality. □

Direct pf of (I) (Weyl): $d=1$, $f \in \mathcal{S}$.

$$\int_{\mathbb{R}} |f|^2 = -\int_{\mathbb{R}} x(|f|^2)'$$

IBP: $\varphi = |f|^2 = f\bar{f} \in \mathcal{S}$

$$\int_{\mathbb{R}} \varphi = -\int_{\mathbb{R}} x\varphi'$$

$$= -\int_{\mathbb{R}} x(f'\bar{f} + \bar{f}' f)$$

$$= -\int_{\mathbb{R}} x \cdot 2\,\mathrm{Re}(f'\bar{f})$$

$$\leq 2\int_{\mathbb{R}} |x|\,|f'\bar{f}|$$

$$\leq 2\left(\int_{\mathbb{R}} x^2|f|^2\right)^{1/2} \cdot \left(\int_{\mathbb{R}} |f'|^2\right)^{1/2}$$

$$= 2\left(\int_{\mathbb{R}} x^2|f|^2\right)^{1/2} \cdot \left(\int_{\mathbb{R}} |2\pi i\xi\hat{f}|^2\right)^{1/2}$$

$$= 4\pi \|xf\|_2 \cdot \|\xi\hat{f}\|_2.$$

$\square$

Ex: Extend to arbitrary d (and $f \in \mathcal{S}$).

Hint: Start with $\sum_{j=1}^{d} x_j \partial_j |f|^2 = 2\,\mathrm{Re}(\nabla f \cdot x\bar{f})$.

If time permits: quantum mechanics origin:

- state of a single particle in $\mathbb{R}^d$ $(d=3)$ is described by

  $\psi : \mathbb{R}^d \to \mathbb{C}$, $\|\psi\|_{L^2} = 1$, – wave f-n,

  ψ and $C\psi$, $|C|=1$, describe the same state

  $|\psi|^2 dx$ prob. of finding a particle in a region:

  $$P[\text{particle lies in } A] = \int_A |\psi|^2$$

  mean position: $\bar{x} = \int_{\mathbb{R}^d} x |\psi|^2 \quad (\bar{x}_j = \int_{\mathbb{R}^d} x_j |\psi|^2)$

  variance $= \int_{\mathbb{R}^d} |x-\bar{x}|^2 |\psi|^2 \quad (=: Var(\psi))$

  – uncertainty in measuring position

Ex: $Var(\psi) = \inf_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} |x-y|^2 |\psi(x)|^2 dx$

- Observables: self-adjoint operators in a Hilbert space – in our case, in $L^2(\mathbb{R}^d)$

  $\langle A\psi, \psi \rangle$ expectation of the observable corresp. to A on a particle described by ψ – average continuation

value of measure for the observable on a collection of identical particles described by ψ.

$\bar{A}\psi = \langle A\psi, \psi \rangle$

$Var(A) = \| A\psi - \bar{A}\psi \|^2$ - the variance/uncertainty in measurement of A.

observable "position" $A\psi = x\psi$ ("vector of op-rs")

"momentum" $A\psi = \frac{1}{2\pi i} \cdot \frac{\partial}{\partial x}\psi$

mean $\int_{\mathbb{R}^d} \xi |\hat{\psi}|^2$

variance $Var(\hat{\psi})$.

$\Rightarrow \quad Var(\psi) \cdot Var(\hat{\psi}) \geq \left(\frac{d}{4\pi}\right)^2.$

- Extension to arbitrary $f \in L^2$ is more tedious.

$d=1$

Wlog, $\|\xi \hat{f}\|_2, \|x\hat{f}\|_2 < \infty$ ($\Rightarrow f, \hat{f}$ are defined pointwise)

Put $f' := [2\pi i \xi \hat{f}]^\vee$, $f' \in L^2$ ($'$ is a symbol here)

- We need to show that $\int_{\mathbb{R}} x(f'\bar{f} + \bar{f'} f) = -\|f\|_2^2$;

the rest is the same.

Ex: Complete the proof. Show that equality sign attains for $f(x) = Ce^{-ax^2}$, $a>0$, and only for them.

Hint: Approximate f by a sequence $(f_n) \subset S$ s.t.

$\int_{\mathbb{R}} (1+4\pi\xi^2)|\hat{f}_n - \hat{f}|^2 \to 0$ for $n \to \infty$.

Wiener uncertainty

Thm (Hardy): $f(x)=O(e^{-\pi x^2})$, $\hat{f}(\xi)=O(e^{-\pi\xi^2}) \Rightarrow f=Ce^{-\pi x^2}$

Ex: "formal extension": $f(x)=O(e^{-ax^2})$, $\hat{f}(\xi)=O(e^{-b\xi^2})$.

$ab>\pi^2 \Rightarrow f=0$

$ab=\pi^2 \Leftrightarrow f=Ce^{-ax^2}$, $\hat{f}=Ce^{-b\xi^2}$.

$ab<\pi^2$ infinite-dimens. linear space.

A piece of complex analysis:

Phragmén–Lindelöf principle:

Warm-up: f analytic in a half-plane Π, continuous in $\overline{\Pi}$, and bdd. Suppose $|f|\le 1$ on $\partial\Pi$.

Then $|f|\le 1$ in Π.

Pf: Wlog, $\Pi=\{Re(z)>0\}$. Put $f_\varepsilon(z)=\frac{f(z)}{(1+z)^\varepsilon}$. Fix $z\in\Pi$.

Take $R\gg|z|$, $\Pi_R=\Pi\cap R\mathbb{D}$, $z\in\Pi_R$.

$|f_\varepsilon|\le 1$ on $\partial\Pi_R \Rightarrow |f_\varepsilon|\le 1 \Rightarrow |f_\varepsilon(z)|\le|1+z|^\varepsilon$. Then $\varepsilon\downarrow 0$

$\square$

Thm (Phragmén-Lindelöf). S sector of opening $<\pi$.

$f \in Hol(S) \cap C(\bar{S})$, $|f(z)| \le A e^{B|z|}$, $z \in S$.

Suppose $|f| \le 1$ on ∂S. Then $|f| \le 1$ in S.

Pf: Wlog, $S = \{|\arg(z)| < \alpha\}$, $\alpha < \frac{\pi}{2}$.

Fix $C > 1$ s.t. $\alpha C < \frac{\pi}{2}$, $\varepsilon > 0$.

Consider $f_\varepsilon(z) = f(z) e^{-\varepsilon z^C}$, $z \in S$

$|f_\varepsilon(z)| \le |f(z)| e^{-\varepsilon \operatorname{Re}(z^C)}$ $\qquad \theta = \arg(z)$

$\le A e^{B|z| - \varepsilon |z|^C \cos C\theta}$ ≤ 1 if $|z| = R \gg 1$.

On ∂S: $|f_\varepsilon| \le 1 \Rightarrow |f| \le 1$ in S.

$\Rightarrow |f(z)| \le e^{\varepsilon |z|^C \cos C\theta}$, $\varepsilon \downarrow 0$ $\qquad \square$

"Formal extension": S sector of opening α

$f \in Hol(S) \cap C(\bar{S})$, $|f(z)| \le A e^{B|z|^\rho}$, $z \in S$, with $\rho < \frac{\pi}{\alpha}$.

Suppose $|f| \le 1$ on ∂S. Then $|f| \le 1$ in S.

Pf of Hardy's thm: Wlog C (implicit const in O) $= 1$.

- f decays super-exponentially

Ex: $f \in L^2(\mathbb{R})$, $|f(x)| \lesssim e^{-a|x|}$
$\Rightarrow \hat{f}$ is analytic in $\{|\mathrm{Im}(\xi)| < \frac{a}{2\pi}\}$

$\Rightarrow \hat{f}(\xi + i\eta) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x + 2\pi \eta x} dx$ analytic in $\mathbb{C}$

$|\hat{f}(\xi + i\eta)| \le \int_{\mathbb{R}} e^{-\pi x^2 + 2\pi \eta x} dx = e^{\pi \eta^2} \int_{\mathbb{R}} e^{-\pi (x+\eta)^2} dx = e^{\pi \eta^2}$

$F(\zeta) := e^{\pi \zeta^2} \hat{f}(\zeta)$,

- $|F(i\eta)| \le 1$
- $|F(\xi)| \le 1$ (assumption)
- $|F(\zeta)| \le e^{\pi(\xi^2 - \eta^2)} |\hat{f}(\zeta)| \le e^{\pi \xi^2}$ everywhere in $\mathbb{C}$.

$\underset{??}{\Rightarrow} |F| \le 1$ in $\mathbb{C} \Rightarrow F = \text{const}$.

We show that $|F| \le 1$ in the 1-st quadrant.

Fix $0 < \beta < \frac{\pi}{2}$, $S_\beta = \{0 < \arg(\zeta) < \beta\}$.

Fix $\delta > 0$.

$$|e^{i\delta \zeta^2} F(\zeta)| \le \begin{cases} e^{\pi \xi^2 - 2\delta \xi \eta} \le 1 & \text{on } \arg(\zeta) = \beta \\ \le 1 & \text{on } \mathbb{R} \end{cases}$$

(β close to $\frac{\pi}{2}$, depending on δ, $\tan \beta > \frac{\pi}{2\delta}$)

$\underset{\text{Phr-Lind}}{\Longrightarrow} |e^{i\delta \zeta^2} F(\zeta)| \le 1$ in S_β.

$\Rightarrow \le 1$ in the first quadrant $(\beta \uparrow \frac{\pi}{2})$

$\Rightarrow |F| \le 1$ in the first quadrant $(\delta \downarrow 0)$ $\square$

Ex: Prove a "polynomial version" of Hardy's thm:

$|f(x)| \lesssim (1+|x|^n)e^{-\pi x^2}$, $|\hat{f}(\xi)| \lesssim (1+|\xi|^2)^n e^{-\pi \xi^2}$

$\Rightarrow f = Pe^{-\pi x^2}$, $\hat{f} = Qe^{-\pi x^2}$, $\deg P, \deg Q \leq n$.

Ex: Prove a "one-sided version" of Hardy's thm:

Suppose that $|f(x)| \lesssim e^{-ax^2}$, $x \geq 0$, and $|\hat{f}(\xi)| \lesssim e^{-b\xi^2}$ for $\xi \geq 0$. Then $f=0$ provided that $a \cdot b > c_0$, where c_0 is a positive numerical constant.

Hint:

- Wlog, $f \in \mathcal{S}$ (otherwise, convolve f with a test function $\chi \in C_0^\infty(\mathbb{R})$, then do the same on the Fourier side)
- Consider the function $(f \cdot \bar{\tilde{f}}) * (\bar{f} \cdot \tilde{f})$, and estimate its (two-sided) decay, $\tilde{f}(x) = f(-x)$,

This argument doesn't give a sharp value of the constant c_0.

The sharp value $c_0 = \pi^2$ is the same as in "the two-sided Hardy theorem". The proof follows similar lines, just at the last step one needs to replace the Phragmén-Lindelöf principle by the following uniqueness lemma, which follows from that principle:

Lemma: Let S be an angle of opening α. Suppose $f \in \mathrm{Hol}(S) \cap C(\bar{S})$,

$$|f(z)| \leq A e^{B|z|^{\pi/\alpha}}, \qquad z \in S,$$

and

$$|f(z)| \leq a e^{-b|z|^{\pi/\alpha}}, \qquad z \in \partial S.$$

Then $f \equiv 0$.

Ex: Prove the lemma and deduce a one-sided version of Hardy's theorem with $c_0 = \pi^2$.

Uniqueness / Non-uniqueness pairs

$\mathcal{H}$ space of functions on $\mathbb{R}$

def $(\Lambda, \Gamma) \subset \mathbb{R} \times \mathbb{R}$ is a uniqueness pair (UP) for $\mathcal{H}$

if: $f \in \mathcal{H}$, $f|_\Lambda = 0$, $\hat{f}|_\Gamma = 0 \Rightarrow f = 0$.

Otherwise, (Λ, Γ) is a non-uniqueness pair (NUP)

• Invariance: $(\Lambda, \Gamma) \mapsto (t\Lambda, \frac{1}{t}\Gamma)$, $t \neq 0$

$(\Lambda, \Gamma) \mapsto (\Lambda + s, \Gamma + \tau)$, $s, \tau \in \mathbb{R}$

Classical examples:

(a) $\Lambda = c\mathbb{Z}$, $\Gamma = \mathbb{R} \setminus (-\frac{1}{2}, \frac{1}{2})$, $\mathcal{H} = L^2$

$c \leq 1$ UP – follows from the sampling thm

$c > 1$ NUP: $f(x) = \sin \pi x \dfrac{\sin \varepsilon x}{x}$

$f|_{\mathbb{Z}} = 0$, $\operatorname{spt}(\hat{f}) = [-\frac{1}{2} - \frac{\varepsilon}{2\pi}, \frac{1}{2} + \frac{\varepsilon}{2\pi}]$. (Then rescale f)

(b) (Benedicks): $f \in L^1(\mathbb{R}^d)$

$f, \hat{f}$ have spts of finite meas. $\Rightarrow f = 0$

Preliminaries (for the pf of Benedicks thm):

• Periodization: $Pf(x) = \sum_{k\in\mathbb{Z}^d} f(x+k)$, $x \in \mathbb{T}^d$.

Lemma / Exercise: Suppose $f \in L^1(\mathbb{R}^d)$. Then

(a) the series converges in $L^1(\mathbb{T}^d)$,

(b) $$\int_{\mathbb{R}^d} f = \int_{\mathbb{T}^d} Pf,$$

(c) $\widehat{Pf}(k) = \hat{f}(k)$, $k \in \mathbb{Z}^d$.

Remark: (c) is a form of the Poisson summation.

Pf of Benedicks thm: $A = \{f \neq 0\}$, $B = \{\hat{f} \neq 0\}$.

Wlog, $m(A) < 1$ (after a rescaling)

• $\int_{\mathbb{T}^d} P\mathbb{1}_A = \int_{\mathbb{R}^d} \mathbb{1}_A = m(A) < 1$

$\Rightarrow \exists X \subset \mathbb{T}^d$, $m(X) > 0$ s.t. $P\mathbb{1}_A = 0$ on X

$\Rightarrow$ for $x \in X$, $f(x+k) = 0$, $\forall k \in \mathbb{Z}^d$ (α)

• $\int_{\mathbb{T}^d} P\mathbb{1}_B = m(B) < \infty$

$\Rightarrow \exists Y \subset \mathbb{T}^d$, $m(Y) = 1$ s.t. $P\mathbb{1}_B < \infty$ on Y

$\Leftrightarrow$ for $\xi \in Y$, $\xi + k \in B$ only for finitely many k's

$\Rightarrow$ for $\xi \in Y$, $\hat{f}(\xi + k) \neq 0$ ONLY for finitely many k's. (β)

Fix $\xi \in \mathbb{T}^d$, consider

$$f_\xi(x) = \sum_{k \in \mathbb{Z}^d} f(x+k) e^{-2\pi i \xi \cdot (x+k)}$$

- $f \in L^1(\mathbb{R}^d) \Rightarrow f_\xi \in L^1(\mathbb{T}^d)$ (by (a))
- $\hat{f_\xi}(k) = \hat{f}(k+\xi)$ (by (c))

$\underset{(\beta)}{\Rightarrow}$ for a.e. $\xi \in \mathbb{T}^d$, f_ξ is a trig. polynomial

- $(\alpha) \Rightarrow f_\xi$ vanishes on X, $m(X) > 0$

Ex: $\Rightarrow f_\xi = 0$ (hint: use that f_ξ is a trig. polynomial)

$\Rightarrow \hat{f}(k+\xi) = 0 \quad \forall k \in \mathbb{Z}^d$, for a.e. $\xi \in \mathbb{T}^d$

$\Rightarrow \hat{f} = 0$

$\Rightarrow f = 0$ $\square$

Remark/Ex: Check that the pf only uses that

- for a.e. $x \in \mathbb{T}^d$, the set $\operatorname{spt}(f) \cap (x + \mathbb{Z}^d)$ is finite;
- for a.e. $\xi \in \mathbb{T}^d$, the set $\operatorname{spt}(\hat{f}) \cap (\xi + \mathbb{Z}^d)$ is finite.

Ex: Prove a quantitative version of Benedicks' theorem (Amrein-Berthier): $m(A), m(B) < \infty$

$$\Rightarrow \quad \|f\|_2^2 \leq C(A,B)\left(\int_{\mathbb{R}^d \setminus A} |f|^2 + \int_{\mathbb{R}^d \setminus B} |\hat{f}|^2\right), \quad f \in L^2(\mathbb{R}^d)$$

<u>Hint:</u>

- It suffices to show that $\|f\|_2^2 \leq C_1(A,B) \int_{\mathbb{R}^d \setminus A} |f|^2$, provided that $\operatorname{spt}(\hat{f}) \subset B$.
- Suppose $\exists\, (f_n) \subset L^2(\mathbb{R}^d)$, $\|f_n\|_{L^2} = 1$, $\operatorname{spt}(\hat{f}_n) \subset B$, while $\int_{\mathbb{R}^d \setminus A} |f_n|^2 \to 0$. Wlog, $(\hat{f}_n)$ is weakly convergent in L^2, denote by $\hat{f}$ its weak limit.
  - Show that $f_n \to f$ pointwise, au then that $f_n \mathbb{1}_A \to f$ in L^2.
  - Check that $\|f\| = 1$, $\operatorname{spt}(f) \subset A$, $\operatorname{spt}(\hat{f}) \subset B$, which contreadicts Benedick's theorem.

<u>Remark</u>: Nazarov showed that

$$C(A,B) \leq e^{C\, m(A)\, m(B)}.$$

• Radchenko–Viazovska (2019)

$\Lambda = \Gamma = \{0, \sqrt{1}, \sqrt{2}, \dots\}$, $\mathcal{H} = \mathcal{S}_{even}$ is a UP

[in the general case one needs to add $f'(0), \hat{f}'(0)$].

This follows from their interpolation formula, which expresses any function $f \in \mathcal{S}$ by the values of f and $\hat{f}$ on the set $\{0, \pm\sqrt{1}, \pm\sqrt{2}, \pm\sqrt{3}, \dots\}$ and two more values $f'(0)$ and $\hat{f}'(0)$. Radchenko and Viazovska approach was based on the theory of modular forms.

Very recently, Gabrielli–Gauthier and Venkatesh found a representation theory approach to their results.

• Toy result: "uniformly super-critical pairs"

$\Lambda = \{ \dots < \lambda_{j-1} < \lambda_j < \lambda_{j+1} < \dots \}, \quad \lambda_j \to \pm\infty$

$\Gamma = \{ \dots < \gamma_{j-1} < \gamma_j < \gamma_{j+1} < \dots \}, \quad \gamma_j \to \pm\infty \qquad (j \to \pm\infty)$

Thm Suppose

$$\max_{j\in\mathbb{Z}} (|\lambda_j|, |\lambda_{j+1}|)(\lambda_{j+1} - \lambda_j) < \frac{1}{2},$$

$$\max_{j\in\mathbb{Z}} (|\gamma_j|, |\gamma_{j+1}|)(\gamma_{j+1} - \gamma_j) < \frac{1}{2}.$$

Then (Λ, Γ) is a UP for $\mathcal{S}$.

Pf: Take $a<1$ s.t. LHS $< \frac{a}{2}$. Then

$$\int_{\mathbb{R}} x^2 |f|^2 = \sum_j \int_{\lambda_j}^{\lambda_{j+1}} x^2 |f|^2$$

$$\leq \sum_j \max(|\lambda_j|, |\lambda_{j+1}|)^2 \int_{\lambda_j}^{\lambda_{j+1}} |f|^2$$

$$\leq \sum_j \max(|\lambda_j|, |\lambda_{j+1}|)^2 \left(\frac{\lambda_{j+1} - \lambda_j}{\pi}\right)^2 \int_{\lambda_j}^{\lambda_{j+1}} |f'|^2$$

$$\leq \frac{a^2}{4\pi^2} \sum_j \int_{\lambda_j}^{\lambda_{j+1}} |f'|^2 = a^2 \int_{\mathbb{R}} \xi^2 |\hat{f}|^2.$$

• In the same way, $\int_{\mathbb{R}} \xi^2 |\hat{f}|^2 \leq a^2 \int_{\mathbb{R}} x^2 |f|^2$

$a < 1 \Rightarrow f = 0.$ $\square$

Ex: The result persists for $a=1$.

Hint: Use description of extremal functions in the Wirtinger inequality.

Remark: the proof uses only f' and $\hat{f}'$ so the result holds for a much wider space $\mathcal{H}$:

Sobolev space:

$$H_1 = \{ f \in L^2 : \xi \hat{f} \in L^2 \}, \quad \|f\|_{H_1}^2 = \int_{\mathbb{R}} (1+\xi^2) |\hat{f}|^2$$

$$\mathcal{H} = \{ f : f, \hat{f} \in H_1 \}, \quad \|f\|_{\mathcal{H}}^2 = \|f\|_{H_1}^2 + \|\hat{f}\|_{H_1}^2$$

def A sequence $\Lambda = (\lambda_j)_{j \in \mathbb{Z}}$ is separated if

$$\lambda_{j+1} - \lambda_j \gtrsim \frac{1}{1 + \min(|\lambda_j|, |\lambda_{j+1}|)}$$

Ex: Suppose (Λ, Γ) is a separated uniformly super-critical pair. Then there exist constants $A, a > 0$ s.t. $\forall f \in \mathcal{H}$

$$a \|f\|_{\mathcal{H}}^2 \leq \sum_{\lambda \in \Lambda} (1+|\lambda|)|f(\lambda)|^2 + \sum_{\gamma \in \Gamma} (1+|\gamma|)|\hat{f}(\gamma)|^2 \leq A \|f\|_{\mathcal{H}}^2 .$$

Hint: to show the left inequality use a quantitative version of the Wirtinger inequality. The right inequality follos from an elementary bound: for $a < b$,

$$|f(a)|^2 \leq C\left[\frac{1}{b-a} \int_a^b |f|^2 + (b-a) \int_a^b |f'|^2\right].$$

def: A discrete set $\Gamma \subset \mathbb{R}$ is ℓ-dense if $\mathbb{R} \setminus \Gamma$ contains no interval of length greater than ℓ.

Ex:

(a) Let $t > 0$ and let Γ be a $(2t)^{-1}$-dense discrete subset of $\mathbb{R}$. Then, for any $\alpha \geq 1$,

$$f \in \mathcal{H},\ f\big|_{\Gamma} = 0 \Rightarrow t^{2\alpha} \int_{\mathbb{R}} |f|^2 \leq \int_{\mathbb{R}} |\xi|^{2\alpha} |\hat{f}(\xi)|^2 .$$

(b) Let Γ be a π-dense discrete subset of $\mathbb{R}$. Then, for any $n \in \mathbb{N}$,

$$f \in \mathcal{S},\ f\big|_{\Gamma} = 0 \Rightarrow \int_{\mathbb{R}} |f|^2 \leq \int_{\mathbb{R}} |f^{(n)}|^2 .$$

Suggested reading:

Survey papers:

1. V. Havin, On the uncertainty principle in harmonic analysis. In: Twentieth century harmonic analysis – a celebration, 3-29 Kluwer, 2001.

2. V. Havin, B. Jöricke, The uncertainty principle in harmonic analysis, 177-259. In: Commutative harmonic analysis III. Encyclopedia Math. Sci. 72, Springer, 1995.

3. G. B. Folland, A. Sitaram, The uncertainty principle: a mathematical survey. J. Fourier Anal. & Appl. 3:1 (1997), 207-238.

4. D. Slepian, Some comments on Fourier analysis, uncertainty and modelling. SIAM Rev. 25 (1983), 379-393.

5. H. Landau, An overview of time and frequency limiting. In Fourier techniques and applications (Edited by J. Price), 201-220. Plenum Press, N.Y. 1985.

6. A. Wigderson, Y. Wigderson, The uncertainty principle: variations on a theme. Bull. Amer. Math. Soc. 58 (2021), 225-261; arXiv 2020. - mostly about discrete versions

Book:

A. Olevskii, A. Ulanovskii, Functions with disconnected spectrum. Sampling, interpolation, translates. University lecture series, vol. 65. Amer. math. soc. 2016.

Original papers:

1. F. Nazarov, Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type, Algebra & Analysis 5:4 (1993), 3-66. English translation in St. Petersburg Math. J. 5:4 (1994), 663-717.

2. D. Radchenko, M. Viazovska, Fourier interpolation on the real line, Publ. IHES 129 (2019), 51-81; arXiv 2017.

3. A. Kulikov, F. Nazarov, M. Sodin, Fourier uniqueness and non-uniqueness pairs, arXiv 2023.