On discrete, continuous and arithmetic aspects of Fourier uncertainty

Alex Iosevich

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Dedication

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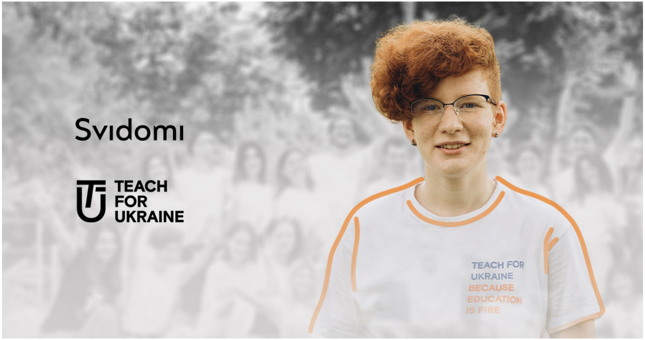

Yuliia Zdanovska 2000-2022

Restriction Conjecture

Conjecture

(Restriction conjecture) The restriction conjecture says that if S^{d-1} is the unit sphere, then

$$\left(\int_{S^{d-1}} \left|\widehat{f}(\xi)\right|^r d\sigma_S(\xi)\right)^{\frac{1}{r}} \leq C_{p,r}\left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{\frac{1}{p}}$$

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- This conjecture is solved in two dimensions and in spite of a lot of brilliant work by Bourgain, Guth, Ou, Stein, Tao, Tomas, Wang and many others, the problem is still open in higher dimensions.

A signal recovery perspective on restriction

- Suppose that A is a compact set in $\mathbb{R}^d$, $d \geq 2$, $|A| > 0$, and $\widehat{1_A}(\xi)$ is known except for $\xi \in S^\delta$, the annulus of radius 1 and thickness δ (small). Can we recover $1_A(x)$ exactly?

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$$= \int_{\xi \notin S^\delta} + \int_{\xi \in S^\delta} = I(x) + II(x).$$
- By assumption, we have no information about $II(x)$, so we must estimate it and hope for the best.

Applying the conjectured restriction inequality

- By Holder, if the restriction theorem holds with exponents (p, r), then

$$|II(x)| \le |S^\delta| \cdot \left(\frac{1}{|S^\delta|} \int_{S^\delta} |\widehat{1}_A(\xi)|^r d\xi \right)^{\frac{1}{r}} \le C_{p,r} \cdot |S^\delta| \cdot |A|^{\frac{1}{p}}.$$

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- If the right hand side is $< \frac{1}{2}$, i.e if $|A| \lesssim \delta^{-p}$ with suitable constants, then we can take the modulus of $I(x)$ and round it up to 1, or down to 0, whichever is closer, and thus recover $1_A(x)$ is exactly.

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- For any r, the restriction theorem always holds for $p = 1$, but according to the restriction conjecture, it holds for any

$$p < \frac{2d}{d+1},$$

which gives us a much less stringent recovery condition.

Finite Signals and Discrete Fourier transform

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- Suppose that the Fourier transform of f is transmitted, where

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- Fourier Inversion says that we can recover the signal by using the Fourier inversion:

$$f(x) = N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \widehat{f}(m).$$

Exact recovery problem

- The basic question is, can we recover f **exactly** from its discrete Fourier transforms if

$$\left\{ \widehat{f}(m) : m \in S \right\}$$

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- The answer turns out to be $\boxed{YES}$ if f is supported in $E \subset \mathbb{Z}_N^d$, and

$$|E| \cdot |S| < \frac{N^d}{2},$$

with the main tool being the Fourier Uncertainty Principle.

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and

- (Plancherel)

$$\sum_{m \in \mathbb{Z}_N^d} \left|\widehat{f}(m)\right|^2 = \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2.$$

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$$= \sum_{y \in \mathbb{Z}_N^d} f(y) N^{-d} \sum_{m \in \mathbb{Z}_N^d} \chi((x-y) \cdot m) = f(x)$$

  by orthogonality.

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A few simple calculations: the paraboloid

- Let N be an odd prime and define

$$P = \{x \in \mathbb{Z}_N^d : x_d = x_1^2 + \cdots + x_{d-1}^2\}.$$

We have

$$\widehat{1}_P(m) = N^{-\frac{d}{2}} \sum_{y \in \mathbb{Z}_N^{d-1}} \chi(-y \cdot m' + ||y|| m_d),$$

where

$$||y|| = y_1^2 + y_2^2 + \cdots + y_{d-1}^2.$$

Paraboloid (continued)

- Suppose that $m_d = 0$ and $m' \neq \mathbf{0}$. Then

$$\widehat{1}_P(m',0) = N^{-\frac{d}{2}} \sum_{y \in \mathbb{Z}_N^{d-1}} \chi(-y \cdot m) = 0.$$

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- If $m_d \neq 0$, let's consider the case $m' \equiv 0$. We obtain

$$N^{-\frac{d}{2}} \sum_{y \in \mathbb{Z}_N^{d-1}} \chi(-m_d ||y||),$$

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- $$g(a) = \sum_{t \in \mathbb{Z}_N} \chi(at^2), \text{ the classical Gauss sum.}$$

Gauss sum estimation

- Suppose that N is an odd prime and $a \neq 0$. We have

$$|g(a)|^2 = \sum_{t,s} \chi(a(t^2 - s^2)) = \sum_{t,s} \chi(ats)$$

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- $$= \sum_{u} \sum_{ts=u} \chi(au) = \sum_{u} \chi(au)n(u),$$

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where
$$n(u) = |\{(t,s) : ts = u\}|.$$
- It is not difficult to see that $n(0) = 2N - 1$ and $N - 1$ otherwise, so
$$|g(a)|^2 = 2N - 1 + (N-1) \sum_{u \neq 0} \chi(au)$$
$$= N + (N-1) \sum_{u} \chi(au) = N.$$

Back to the paraboloid

- It follows that if $a \neq 0$,
$$|g(a)| = \sqrt{N}.$$

Going back to the paraboloid and N is an odd prime, we see that if $m' = \mathbf{0}$, $m_d \neq 0$,

$$|\widehat{1}_M(0, \ldots, 0, m_d)| = N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^{d-1}} \chi(m_d ||y||)$$

$$= N^{-\frac{d}{2}} (\sqrt{N})^{d-1} = N^{-\frac{1}{2}}.$$

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- If $m_d \neq 0$ and $m' \neq (0,\ldots,0)$, we can complete the square and obtain the same bound, i.e
$$|\widehat{1}_P(m)| = N^{-\frac{1}{2}}.$$

The sphere: life becomes much more interesting!

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$$S = \{x \in \mathbb{Z}_N^d : x_1^2 + x_2^2 + \cdots + x_d^2 = 1\}, N \text{ odd prime.}$$

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- Since

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we can change variables above and arrive at

- $$N^{-\frac{d}{2}-1} \sum_{s \neq 0} \sum_{x \in \mathbb{Z}_N^d} \chi(s||x||)\chi(-s)\chi(-||m||/4s).$$

The sphere (continued)

- Using the Gauss sum identity we obtain a few minutes ago, the expression above equals

$$N^{-1}\sum_{s\neq 0}\gamma^d(s)\chi(-s-||m||/4s),$$

where

$$|\gamma(s)|=1.$$

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- The "innocent" looking expression above is a twisted Kloosterman sum. Its modulus is bounded by $2\sqrt{N}$. The proof of this fact is very sophisticated and uses highly non-trivial number theory.
- In conclusion, if $m \neq \mathbf{0}$,

$$|\widehat{1}_S(m)| \leq CN^{-\frac{1}{2}}.$$

The square root law

- In both the case of the sphere and the paraboloid, we established an estimate of the form

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- An interesting situation arises if we ask whether such estimate can ever hold in a non-field setting. The is where we now (briefly) turn our attention.

From Fourier decay to additive energy

- Suppose that S satisfies

$$|\widehat{1}_S(m)| \leq C_{Fourier} N^{-\frac{d}{2}} \cdot |S|^{\frac{1}{2}} \text{ for } m \neq \mathbf{0}.$$

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- We have $\sum_m |\widehat{1}_S(m)|^4 =$

$$= N^{-2d} \sum_{x,y,x',y} \sum_m \chi(m \cdot (x + y - x' - y')) 1_S(x) 1_S(y) 1_S(x') 1_S(y')$$

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From Fourier decay to additive energy (continued)

- By assumption, the right-hand side is bounded by

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From Fourier decay to additive energy (continued)

- By assumption, the right-hand side is bounded by

$$N^d \cdot C^2_{Fourier} \cdot N^{-d} \cdot |S| \cdot \sum_z |\widehat{1}_S(m)|^2.$$

- By Plancherel, this expression equals

$$C^2_{Fourier} \cdot |S|^2,$$

from which we conclude that

$$\frac{\Lambda(S)}{|S|^2} \leq C^2_{Fourier}.$$

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- Since we know nothing about S, the best we can do is assume that the quantity above is small.

An elementary point of view: rounding

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$$N^{-d}|E||S| < \frac{1}{2},$$
we can take the modulus of $I(x)$ and round it up to 1 if it is $\geq \frac{1}{2}$, and round it down to 0 otherwise.

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- This gives us **exact recovery** using a simple and direct algorithm (to be henceforth referred to as the Direct Rounding Algorithm (DRA)) if

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- But what happens if we consider general signals?

Matolcsi-Szucks/ Donoho-Stark point of view

- Let $h : \mathbb{Z}_N^d \to \mathbb{C}$. Then the classical Uncertainty Principle says that

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- If f cannot be recovered uniquely, then there exists a signal $g : \mathbb{Z}_N^d \to \mathbb{C}$ such that g also has $|supp(f)|$ non-zero entries,

$$\widehat{f}(m) = \widehat{g}(m) \text{ for } m \notin S,$$

and f is not identically equal to g.

Uncertainty Principle $\to$ Unique Recovery

- Let $h = f - g$. It is clear that $\widehat{h}$ has at most $|S|$ non-zero entries, and h has at most $2|supp(f)|$ non-zero entries.

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- By the Uncertainty Principle, we must have

$$|supp(f)| \cdot |S| \geq \frac{N^d}{2}.$$

- Therefore, if we assume that

$$|supp(f)| \cdot |S| < \frac{N^d}{2},$$

we must have $h = 0$, and hence the recovery is *unique*.

The classical uncertainty principle is, in general, sharp

- Let N be an odd prime, and let S be a k-dimensional subspace of $\mathbb{Z}_N^d$, $1 \leq k \leq d-1$.

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- We are going to see that in the presence of non-trivial restriction estimates, we can do much better. We are also going to see that non-trivial restriction estimates "typically" hold.

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- Summing both sides over $x \in E$ and cancelling the L^1 norms of h on both sides, we obtain
$$|E| \cdot |S| \geq N^d.$$

Restriction theory enters the picture

- We say that $S \subset \mathbb{Z}_N^d$ satisfies the (p,q) restriction estimate $(1 \leq p \leq q)$ with uniform constant $C_{p,q} > 0$ if for any function $f : \mathbb{Z}_N^d \to \mathbb{C}$,

$$\left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^q \right)^{\frac{1}{q}} \leq C_{p,q} N^{-\frac{d}{2}} \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^p \right)^{\frac{1}{p}}.$$

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- We have the following "universal" restriction theorem.

Theorem

(A.I. and A. Mayeli) Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ and let S be a subset of $\mathbb{Z}_N^d$. Then

$$\left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^2 \right)^{\frac{1}{2}} \leq \left(\frac{|S|}{N^{\frac{d}{2}}} \right)^{-\frac{1}{2}} \cdot \left(\max_{U \subset S} \frac{\Lambda(U)}{|U|^2} \right)^{\frac{1}{4}} \cdot N^{-\frac{d}{2}} \cdot \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^{\frac{4}{3}} \right)^{\frac{3}{4}}.$$

From restriction directly to uncertainty

- Before proving the universal restricion theorem, we are going to develop a simple mechanism for going directly from restriction to uncertainty, where the more non-trivial the restriction estimate becomes, the better uncertainty principle we obtain. More eleborate versions of this approach will be developed a bit later.

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Theorem (Uncertainty Principle via Restriction Theory – A.I. & A.Mayeli, 2023)

Suppose that $f, \widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$, *with* f *supported in* $E \subset \mathbb{Z}_N^d$, *and* $\widehat{f}$ *supported in* $S \subset \mathbb{Z}_N^d$. *Suppose* S *satisfies the* (p, q) *restriction estimate with norm* $C_{p,q}$. *Then*

$$|E|^{\frac{1}{p}} \cdot |S| \geq \frac{N^d}{C_{p,q}}.$$

Proof of Uncertainty via Restriction

- Suppose that f is supported in a set E, and $\widehat{f}$ is supported in a set S. Then by the Fourier Inversion Formula and the support condition,

$$f(x) = N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \widehat{f}(m) = N^{-\frac{d}{2}} \sum_{m \in S} \chi(x \cdot m) \widehat{f}(m).$$

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- By Holder's inequality,

$$|f(x)| \leq N^{-\frac{d}{2}} \cdot |S| \cdot \left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^q \right)^{\frac{1}{q}}.$$

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- By the restriction bound assumption, this expression is bounded by

$$|S| \cdot C_{p,q} \cdot N^{-d} \cdot \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^p \right)^{\frac{1}{p}},$$

Proof of Uncertainty Principle via Restriction I (continued)

- and by the support assumption, this quantity is equal to

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Proof of Uncertainty Principle via Restriction I (continued)

- and by the support assumption, this quantity is equal to

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- Putting everything together, we see that

$$|f(x)| \leq |S| \cdot C_{p,q} \cdot N^{-d} \cdot \left(\sum_{x \in E} |f(x)|^p \right)^{\frac{1}{p}}.$$

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- Putting everything together, we see that

$$|f(x)| \leq |S| \cdot C_{p,q} \cdot N^{-d} \cdot \left(\sum_{x \in E} |f(x)|^p \right)^{\frac{1}{p}}.$$

- Raising both sides to the power of p, summing over E, and dividing both sides of the resulting inequality by $\sum_{x \in E} |f(x)|^p$, we obtain

$$|S|^p \cdot |E| \cdot C_{p,q}^p \geq N^{dp}.$$

Proof of Uncertainty Principle via Restriction I (finale)

- or, equivalently,

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  as desired.

- This completes the proof of the Uncertainty Principle via Restriction Theory.

Proof of the universal restriction theorem

- We have
$$\sum_{m \in S} |\widehat{f}(m)|^2 = \sum_m 1_S(m)\widehat{f}(m)g(m),$$
where
$$g(m) = \overline{1_S \widehat{f}(m)}.$$

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- The expression above equals
$$\sum_x f(x)\widehat{1_S g}(x) \leq ||f||_{L^{\frac{4}{3}}(\mathbb{Z}_N^d)} \cdot \left(\sum_{x \in \mathbb{Z}_N^d} |\widehat{1_S g}(x)|^4 \right)^{\frac{1}{4}}.$$

Proof of the universal restriction theorem (continued)

- We have

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- The quantity above is bounded by

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- It follows that

$$\left(\sum_{x \in \mathbb{Z}_N^d} |\widehat{1_S g}(x)|^4 \right)^{\frac{1}{4}} \leq N^{-\frac{d}{4}} \cdot \left(\max_{U \subset S} \frac{\Lambda(U)}{|U|^2} \right)^{\frac{1}{4}} \cdot ||g||_{L^2(\mathbb{Z}_N^d)}.$$

Proof of the universal restriction theorem (continued)

- Putting everything together, we see that

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- This completes the proof of the universal restriction theorem.

An additive energy uncertainty principle

- It would be very convenient to work out a version of the additive energy uncertainty principle purely in terms of the additive energy of $E = supp(f)$ and $S = supp(\widehat{f})$. This is where we not turn our attention.

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Theorem

(K. Aldahleh, A. Iosevich, J. Iosevich, J. Jaimangal, A. Mayeli, and S. Pack) Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ with $supp(f) = E$ and $supp(\widehat{f}) = S$. Then for any $\alpha \in [0,1]$,

$$N^d \le \Lambda^{\frac{\alpha}{3}}(E)\Lambda^{\frac{1-\alpha}{3}}(S)|E|^{1-\alpha}|S|^{\alpha}.$$

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- It follows that

$$|f(x)| \leq N^{-\frac{d}{2}} \cdot |S|^{\frac{3}{4}} \cdot \left(\sum_{m \in \mathbb{Z}_N^d} \left| \widehat{f}(m) \right|^4 \right)^{\frac{1}{4}}.$$

Proof of the additive energy uncertainty principle (continued)

- We have

$$\sum_{m \in S} |\widehat{f}(m)|^4$$

$$= N^{-2d} \sum_{m \in \mathbb{Z}_N^d} \sum_{x,y,x',y' \in E} \chi((x+y-x'-y') \cdot m)\overline{f(x)f(y)}f(x')f(y')$$

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- $$= N^{-d} \sum_{x+y=x'+y';x,y,x',y'\in E} \overline{f(x)f(y)}f(x')f(y')$$

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$$\leq N^{-d} \cdot \Lambda(E) \cdot ||f||^4_{L^\infty(E)}.$$

Proof of the additive energy uncertainty principle (continued)

- Putting everything together, we see that

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- Taking the maximum over $x \in E$ and cancelling the $L^\infty(E)$ norms, we obtain

$$N^{\frac{3d}{4}} \leq \Lambda^{\frac{1}{4}}(E) \cdot |S|^{\frac{3}{4}}.$$

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- Reversing the roles of E and S, we obtain

$$N^d \leq \Lambda^{\frac{1}{3}}(S) \cdot |E|, \text{ which completes the proof.}$$

Bourgain's Λ_q theorem - general formulation

- Jean Bourgain proved that if G is a locally compact abelian group, $\phi_1, \dots, \phi_n$ are orthogonal functions with $||\phi_j||_\infty \leq 1$, the for a generic set $S \subset \{1, 2, \dots, n\}$ of size $\approx n^{\frac{2}{q}}$, $q > 2$,

$$\left\| \sum_{i \in S} a_i \phi_i \right\|_{L^q(G)} \leq C(q) \cdot \left(\sum_{i \in S} |a_i|^2 \right)^{\frac{1}{2}},$$

where $C(q)$ depends only on q.

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where $C(q)$ depends only on q.

- As we shall see, this result has a beautiful built-in uncertainty principle.

Bourgain's Λ_q theorem

- It is a consequence of Bourgain's celebrated Λ_q theorem in locally compact abelian groups that if $f : \mathbb{Z}_N^d \to \mathbb{C}$ and $\widehat{f}$ **is supported in** S, then for a "generic" set of size $\approx N^{\frac{2d}{q}}$, $2 < q < \infty$,

$$\left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^q \right)^{\frac{1}{q}} \leq K_q(S) \left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2 \right)^{\frac{1}{2}},$$

with $K_q(S)$ independent of N.

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  with $K_q(S)$ independent of N.

- It is not difficult to see that this inequality implies that the support of f must be a positive proportion of $\mathbb{Z}_N^d$.

A direct consequence of Bourgain's Λ_q theorem

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A direct consequence of Bourgain's Λ_q theorem

- Suppose that S is generic, as in Bourgain's theorem.
- Suppose that f is supported in $E \subset \mathbb{Z}_N^d$ and $\widehat{f}$ is supported in S. Bourgain's theorem implies that
- $$N^{-\frac{d}{q}} \cdot |E|^{\frac{1}{q}} \left(\frac{1}{|E|} \sum_{x \in E} |f(x)|^q \right)^{\frac{1}{q}}$$
$$\leq K_q(S) N^{-\frac{d}{2}} \cdot |E|^{\frac{1}{2}} \left(\frac{1}{|E|} \sum_{x \in E} |f(x)|^2 \right)^{\frac{1}{2}}.$$

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- It follows that

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- It follows that if $\widehat{f}$ is supported in a generic set of size $\approx N^{d-\epsilon}$, then f is supported on a positive proportion of $\mathbb{Z}_N^d$.

- We conclude that if we send the Fourier transform of a signal f supported on a set of size $o(N^d)$, and the frequencies in $S \subset \mathbb{Z}_N^d$ satisfying a Λ_q, $q > 2$, inequality are missing, we can recover f exactly and uniquely with very high probability.

Spectral synthesis in $\mathbb{Z}_N^d$

Theorem

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$, *and let* $S \subset \mathbb{Z}_N^d$. *Then*

$$||f||_{L^\infty(\mathbb{Z}_N^d)} \leq \sqrt{\frac{|S|}{N^{\frac{2d}{p}}}} \cdot ||f||_{L^p(\mathbb{Z}_N^d)},$$

and

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where $\check{f}$ *denotes the inverse Fourier transform of* f.

-

Spectral synthesis in $\mathbb{Z}_N^d$

Theorem

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$*, and let* $S \subset \mathbb{Z}_N^d$*. Then*

$$||f||_{L^\infty(\mathbb{Z}_N^d)} \leq \sqrt{\frac{|S|}{N^{\frac{2d}{p}}}} \cdot ||f||_{L^p(\mathbb{Z}_N^d)},$$

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where $\check{f}$ *denotes the inverse Fourier transform of* f*.*

-
- Observe that if $||f||_{L^\infty(\mathbb{Z}_N^d)} \geq \delta$, say, and $\sqrt{\frac{|S|}{N^{\frac{2d}{p}}}}$ is sufficiently small, then we can conclude that f is identically 0 if $||f||_{L^p(\mathbb{Z}_N^d)}$ is uniformly bounded.

Proof of spectral synthesis in $\mathbb{Z}_N^d$ theorem

- By Fourier inversion and the assumption that $\widehat{f}$ is supported in S,

$$f(x) = N^{-\frac{d}{2}} \sum_{m \in S} \chi(x \cdot m) \widehat{f}(m).$$

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- It follows that

$$|f(x)| \leq N^{-\frac{d}{2}} \cdot |S|^{\frac{1}{2}} \left(\sum_{m \in \mathbb{Z}_N^d} |\widehat{f}(m)|^2 \right)^{\frac{1}{2}}.$$

By Plancherel, this quantity is equal to

$$N^{-\frac{d}{2}} \cdot |S|^{\frac{1}{2}} \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^2 \right)^{\frac{1}{2}}$$

Proof of spectral synthesis in $\mathbb{Z}_N^d$ theorem (continued)

- $$= |S|^{\frac{1}{2}} \left(N^{-d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2 \right)^{\frac{1}{2}}.$$

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- By Holder's inequality, this quantity is bounded by

$$|S|^{\frac{1}{2}} \left(N^{-d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^p \right)^{\frac{1}{p}}$$

Proof of spectral synthesis in $\mathbb{Z}_N^d$ theorem (continued)

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$$\widehat{f}(m) = \widehat{f}(m) 1_S(m).$$

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$$f(x) = N^{-\frac{d}{2}} \cdot f * \check{1}_S(x).$$

- We conclude (by Holder) that
$$|f(x)| \leq N^{-\frac{d}{2}} \cdot ||f||_{L^p(\mathbb{Z}_N^d)} \cdot ||\check{1}_S||_{L^{p'}(\mathbb{Z}_N^d)}.$$

Application to signal recovery

Theorem

Suppose that $f : \mathbb{Z}_N^d \to \mathbb{R}$, *where* $\{f(x) : x \in \mathbb{Z}_N^d\} \subset \delta\mathbb{Z}$. *Suppose that the Fourier transform of* f *is transmitted with the frequencies* $\{\widehat{f}(m)\}_{m \in S}$ *unobserved. Suppose that*

$$|S| = C_{size} N^k.$$

Then f *can be recovered exactly and uniquely if*

$$||f||_{L^{\frac{2d}{k}}(\mathbb{Z}_N^d)} < \frac{\delta}{2\sqrt{C_{size}}}.$$

-

Proof of the signal recovery theorem

- Suppose that we cannot recover f uniquely. Then there exists $g : \mathbb{Z}_N^d$ such that

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$\widehat{f}(m) = \widehat{g}(m)$ outside of S, and f is not identically equal to g.

- Let $h = f - g$. Then
$$||h||_p \leq ||f||_p + ||g||_p \leq 2||f||_p$$
by Minkowski's theorem, and the support of $\widehat{h}$ is contained in S since $\widehat{f}$ and $\widehat{g}$ agree away from S.

Proof of the signal recovery theorem (finale)

- The separation condition on f and g implies that

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- Applying the spectral synthesis in $\mathbb{Z}_N^d$ theorem with $p = \frac{2d}{k}$ and the observations above, we see that

$$\delta \leq ||h||_{\infty} \leq 2||f||_{L^{\frac{2d}{k}}(\mathbb{Z}_N^d)} \cdot \sqrt{C_{size}}.$$

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$$\delta \leq ||h||_\infty \leq 2||f||_{L^{\frac{2d}{k}}(\mathbb{Z}_N^d)} \cdot \sqrt{C_{size}}.$$

- It follows that if we assume that $||f||_{L^{\frac{2d}{k}}(\mathbb{Z}_N^d)} < \frac{\delta}{\sqrt{C_{size}}}$, we obtain a contradiction and conclude that h must be identically 0. This concludes the proof of uniqueness.

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- We may discuss the continuous case in more detail later in these lectures.

- For the moment we immerse ourselves back in the world of finite signals.

Annihilating pairs: Ghobber and Jaming

- Let $f : \mathbb{Z}_N^d \to \mathbb{C}$. Ghobber and Jaming proved in 2011 that if $E, S \subset \mathbb{Z}_N^d$, $|E| \cdot |S| < N^d$, then

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq \left(1 + \frac{1}{1 - \sqrt{\frac{|E||S|}{N^d}}}\right) \cdot \left(||f||_{L^2(E^c)} + ||\widehat{f}||_{L^2(S^c)}\right).$$

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- Observe that this result easily implies the classical uncertainty principle since if f is supported in E, $\widehat{f}$ is supported in S, and

$$|E| \cdot |S| < N^d,$$

then the right hand side of the inequality above is 0. Hence the left hand side is also 0 and the uncertainty principle is established.

Proof of the Ghobber-Jaming result

- We have

$$||\widehat{1_E f}||_{L^2(S)} \le N^{-\frac{d}{2}} \cdot |S|^{\frac{1}{2}} \cdot ||f||_{L^1(E)}$$

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- On the other hand,

$$||\widehat{1_E f}||_{L^2(S^c)} \geq ||\widehat{1_E f}||_{L^2(\mathbb{Z}_N^d)} - ||\widehat{1_E f}||_{L^2(S)}$$

- $$\geq ||f||_{L^2(E)} \left(1 - N^{-\frac{d}{2}} \cdot |S|^{\frac{1}{2}} \cdot |E|^{\frac{1}{2}}\right).$$

Proof of the Ghobber-Jaming result (continued)

- We are almost ready to drive for the finish line. By the triangle inequality,
$$||f||_{L^2(\mathbb{Z}_N^d)} \leq ||f||_{L^2(E)} + ||f||_{L^2(E^c)}$$

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- $$= ||\widehat{f} - \widehat{1_{E^c} f}||_{L^2(S^c)} \cdot \frac{1}{1 - \sqrt{\frac{|E||S|}{N^d}}} + ||f||_{L^2(E^c)}$$

Proof of the Ghobber-Jaming result (continued)

- $$\leq \left(||\widehat{f}||_{L^2(S^c)} + ||f||_{L^2(E^c)} \right) \cdot \frac{1}{1 - \sqrt{\frac{|E||S|}{N^d}}} + ||f||_{L^2(E^c)}$$

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- $$\left(1 + \frac{1}{1 - \sqrt{\frac{|E||S|}{N^d}}} \right) \cdot \left(||f||_{L^2(E^c)} + ||\widehat{f}||_{L^2(S^c)} \right),$$

  and the proof is complete.

Annihilating pairs and structure of sets

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- The following is a recent result due to A.I., P. Jaming and A. Mayeli. Suppose that a (p,q) Fourier restriction estimate holds for $S \subset \mathbb{Z}_N^d$, $1 \le p \le 2 \le q$, with norm $C_{p,q}$. Then

$$||f||_{L^2(\mathbb{Z}_N^d)} \le \left(1 + \frac{1}{1 - \sqrt{\frac{C_{p,q}^2 |E|^{\frac{2-p}{p}} |S|}{N^d}}}\right) \cdot \left(||f||_{L^2(E^c)} + ||\widehat{f}||_{L^2(S^c)}\right),$$

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- provided that

$$|E|^{\frac{2-p}{p}} |S| < \frac{N^d}{C_{p,q}^2}.$$

The case $1 \leq p \leq q \leq 2$

- If $1 \leq p \leq q \leq 2$ and if a (p, q) Fourier restriction estimate holds for S,

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq \left(1 + \frac{|E|^{\frac{1}{2} - \frac{1}{q'}}}{1 - \left(\frac{|S||E|^{\frac{(q'-p)q}{q'p}} C_{p,q}^q}{N^d} \right)^{\frac{1}{q}}} \right) \left(||f||_{L^2(E^c)} + ||\widehat{f}||_{L^2(S^c)} \right),$$

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Proof of the A.I.-Jaming-Mayeli result

- We first handle the case $1 \le p \le 2 \le q$. By the restriction assumption,

$$||\widehat{1_E f}||_{L^2(S)} = |S|^{\frac{1}{2}} ||\widehat{1_E f}||_{L^2(\mu_S)} \le |S|^{\frac{1}{2}} ||\widehat{1_E f}||_{L^q(\mu_S)}$$

$$\le |S|^{\frac{1}{2}} \cdot C_{p,q} N^{-\frac{d}{2}} ||f||_{L^p(E)}$$

by assumption.

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$$\le |S|^{\frac{1}{2}} \cdot C_{p,q} N^{-\frac{d}{2}} ||f||_{L^p(E)}$$

by assumption.

- By Holder's inequality, this quantity is bounded by

$$C_{p,q} |S|^{\frac{1}{2}} N^{-\frac{d}{2}} |E|^{\frac{2-p}{2p}} ||f||_{L^2(E)} = \sqrt{\frac{C_{p,q}^2 |S||E|^{\frac{2-p}{p}}}{N^d}} ||f||_{L^2(E)}.$$

Proof of the A.I.-Jaming-Mayeli result (continued)

- On the other hand,

$$||\widehat{1_E f}||_{L^2(S^c)} \geq ||\widehat{1_E f}||_{L^2(\mathbb{Z}_N^d)} - ||\widehat{1_E f}||_{L^2(S)}$$

$$\geq \left(1 - \sqrt{\frac{C_{p,q}^2 |S||E|^{\frac{2-p}{p}}}{N^d}}\right) ||f||_{L^2(E)}.$$

We are now ready for the conclusion of the proof. We have

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq ||f||_{L^2(E)} + ||f||_{L^2(E^c)}$$

$$\leq \left(1 - \sqrt{\frac{C_{p,q}^2 |S||E|^{\frac{2-p}{p}}}{N^d}}\right)^{-1} ||\widehat{1_E f}||_{L^2(S^c)} + ||f||_{L^2(E^c)}.$$

Proof of the A.I.-Jaming-Mayeli result (continued)

- We are left to unravel the quantity $||\widehat{1_E f}||_{L^2(S^c)}$. We have

$$||\widehat{1_E f}||_{L^2(S^c)} = ||1_{S^c}\widehat{f} - 1_{S^c}\widehat{1_{E^c} f}||_{L^2(\mathbb{Z}_N^d)}$$

$$\leq ||\widehat{f}||_{L^2(S^c)} + ||f||_{L^2(E^c)}.$$

Plugging this back into above, we have

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq$$

$$\leq \left(1 - \sqrt{\frac{C_{p,q}^2 |S||E|^{\frac{2-p}{p}}}{N^d}}\right)^{-1} \left(||\widehat{f}||_{L^2(S^c)} + ||f||_{L^2(E^c)}\right) + ||f||_{L^2(E^c)}$$

and the case $1 \leq p \leq 2 \leq q$ is established.

Proof of the A.I.-Jaming-Mayeli result (continued)

- We now handle the case $1 \leq p \leq q \leq 2$. By assumption, we have

$$||\widehat{1_E f}||_{L^q(S)} \leq |S|^{\frac{1}{q}} C_{p,q} N^{-\frac{d}{2}} ||f||_{L^p(E)}$$

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Lemma (Hausdorff-Young inequality)

Suppose that $f : \mathbb{Z}_N^d \to \mathbb{C}$ and $1 \leq p \leq 2$. Then

$$||\widehat{f}||_{L^{p'}(\mathbb{Z}_N^d)} \leq N^{-\frac{d}{2}\left(\frac{2-p}{p}\right)} ||f||_{L^p(\mathbb{Z}_N^d)}.$$

-

Proof of the A.I.-Jaming-Mayeli result (continued)

- The case $p = 1$ follows by the triangle inequality and the definition of the Fourier transform. The case $p = 2$ is Plancherel. The result follows by Riesz-Thorin interpolation theorem.

Proof of the A.I.-Jaming-Mayeli result (continued)

- The case $p = 1$ follows by the triangle inequality and the definition of the Fourier transform. The case $p = 2$ is Plancherel. The result follows by Riesz-Thorin interpolation theorem.

- Using Hausdorff-Young, we have

$$||\widehat{1_E f}||_{L^q(\mathbb{Z}_N^d)} \geq N^{\frac{d}{2}\left(\frac{2-q}{q}\right)} ||f||_{L^{q'}(E)}$$

Proof of the A.I.-Jaming-Mayeli result (continued)

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- $$\geq N^{\frac{d}{2}\left(\frac{2-q}{q}\right)}|E|^{\frac{1}{2}-\frac{1}{q'}}||f||_{L^2(E)}.$$

Proof of the A.I.-Jaming-Mayeli result (continued)

- Combining, we obtain

$$||f||_{L^2(E)} \leq \frac{||\widehat{1_E f}||_{L^q(S^c)}}{N^{\frac{d}{2}\left(\frac{2-q}{q}\right)}|E|^{\frac{1}{2}-\frac{1}{q'}} - |S|^{\frac{1}{q}}|E|^{\frac{1}{p}-\frac{1}{2}} C_{p,q} N^{-\frac{d}{2}}}.$$

Proof of the A.I.-Jaming-Mayeli result (continued)

- Combining, we obtain

$$||f||_{L^2(E)} \leq \frac{||\widehat{1_E f}||_{L^q(S^c)}}{N^{\frac{d}{2}\left(\frac{2-q}{q}\right)}|E|^{\frac{1}{2}-\frac{1}{q'}} - |S|^{\frac{1}{q}}|E|^{\frac{1}{p}-\frac{1}{2}}C_{p,q}N^{-\frac{d}{2}}}.$$

- We now unravel $||\widehat{1_E f}||_{L^q(S^c)}$. We have

$$||\widehat{1_E f}||_{L^q(S^c)} = ||\widehat{f} - \widehat{1_{E^c} f}||_{L^q(S^c)}$$

Proof of the A.I.-Jaming-Mayeli result (continued)

- Combining, we obtain

$$||f||_{L^2(E)} \leq \frac{||\widehat{1_E f}||_{L^q(S^c)}}{N^{\frac{d}{2}\left(\frac{2-q}{q}\right)}|E|^{\frac{1}{2}-\frac{1}{q'}} - |S|^{\frac{1}{q}}|E|^{\frac{1}{p}-\frac{1}{2}}C_{p,q}N^{-\frac{d}{2}}}.$$

- We now unravel $||\widehat{1_E f}||_{L^q(S^c)}$. We have

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- $$\leq ||\widehat{f}||_{L^q(S^c)} + ||\widehat{1_{E^c} f}||_{L^q(S^c)}$$

- $$\leq |S^c|^{\frac{1}{q}-\frac{1}{2}} \left(||\widehat{f}||_{L^2(S^c)} + ||f||_{L^2(E^c)} \right).$$

Proof of the A.I.-Jaming-Mayeli result (continued)

- $$\leq |S^c|^{\frac{1}{q}-\frac{1}{2}} \left(||\widehat{f}||_{L^2(S^c)} + ||f||_{L^2(E^c)} \right).$$

- We have
$$||f||_{L^2(\mathbb{Z}_N^d)} \leq ||f||_{L^2(E)} + ||f||_{L^2(E^c)}.$$

Proof of the A.I.-Jaming-Mayeli result (continued)

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- Rearranging the terms yields the conclusion of the case $1 \leq p \leq q \leq 2$.

A consequence of annihilating pairs inequalities

- The following result was originally proven directly by A.I. and A. Mayeli earlier this year, but it also follows directly from the annihilating pairs inequalities we just proved.

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Theorem

Suppose that $f : \mathbb{Z}_N^d \to \mathbb{C}$ is supported in $E \subset \mathbb{Z}_N^d$, and $\hat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ is supported in $S \subset \mathbb{Z}_N^d$. Suppose S satisfies the (p, q) restriction estimate with norm $C_{p,q}$, $1 \leq p \leq q$, $p \leq 2$.

i) If $q \geq 2$, then

$$|E|^{\frac{2-p}{p}} \cdot |S| \geq \frac{N^d}{C_{p,q}^2}.$$

ii) If $1 \leq p \leq q \leq 2$, then

$$|E|^{\frac{(q'-p)q}{q'p}} \cdot |S| \geq \frac{N^d}{C_{p,q}^q}.$$

From Restriction to Exact Recovery

Corollary

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ with support $supp(f) = E$. Let r be another signal with support of the same size such that $\widehat{r}(m) = \widehat{f}(m)$ for $m \notin S$, and 0 otherwise. Suppose $S \subset \mathbb{Z}_N^d$ satisfies the (p,q), $p < 2$, restriction estimate with uniform constant $C_{p,q}$. Then f can be reconstructed from r uniquely if

$$|E|^{\frac{1}{p}} \cdot |S| < \frac{N^d}{2^{\frac{1}{p}} C_{p,q}},$$

or if

$$|E|^{\frac{2-p}{p}} \cdot |S| < \frac{N^d}{2^{\frac{2-p}{p}} C_{p,q}^2} \text{ when } q \geq 2,$$

and

$$|E|^{\frac{(q'-p)q}{q'p}} \cdot |S| < \frac{N^d}{2^{\frac{(q'-p)q}{q'p}} C_{p,q}^q} \text{ when } q \leq 2.$$

Concentration inequality

- Donoho and Stark showed that if $f : \mathbb{Z}_N^d \to \mathbb{C}$, and $E, S \subset \mathbb{Z}_N^d$ such that f is concentrated in E at level ϵ_E in the sense that

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  with ϵ_E, ϵ_S both < 1, then
- $$\epsilon_E + \epsilon_S \geq 1 - \sqrt{\frac{|E||S|}{N^d}}.$$

Concentration inequality (continued)

- The following is a direct consequence of our annihilation pairs inequalities.

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Corollary

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ and suppose that f is L^2-concentrated on E at level $\epsilon_E > 0$ and $\widehat{f}$ is L^2-concentrated on S at level ϵ_S. Suppose that $S \subset \mathbb{Z}_N^d$ satisfying the (p,q) restriction estimate with norm $C_{p,q}$. Then

$$\epsilon_E + \epsilon_S \geq \frac{1}{1 + \frac{1}{1 - \sqrt{\frac{C_{p,q}^2 |E|^{\frac{2-p}{p}} |S|}{N^d}}}}.$$

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Concentration inequality (continued)

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- Note that in the case $p = 1$, when the restriction estimate always holds with constant $C_{1,q} = 1$, we recover a condition that is slightly stronger than the Donoho-Stark condition above.

Proof of the concentration inequality

- The concentration inequality and the assumptions on the concentration of f on E and concentration of $\widehat{f}$ on S imply that

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq C_{ann} \left(||f||_{L^2(E^c)} + ||\widehat{f}||_{L^2(S^c)} \right)$$
$$\leq C_{ann}(\epsilon_E + \epsilon_S)||f||_{L^2(\mathbb{Z}_N^d)}.$$

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which implies that

- $$\epsilon_E + \epsilon_S \geq \frac{1}{C_{ann}},$$

and the proof is complete.

Another version of the uncertainty principle

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- Suppose that $f \in L^1_{loc}(\mathbb{R}^d)$ and $\widehat{f}$ is supported in S is a k-dimensional submanifold of $\mathbb{R}^d$. Suppose further that $f \in L^p(\mathbb{R}^d)$ for some $p \leq \frac{2d}{k}$. Then $f \equiv 0$.

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- Suppose that $f \in L^1_{loc}(\mathbb{R}^d)$ and $\widehat{f}$ is supported in S is a k-dimensional submanifold of $\mathbb{R}^d$. Suppose further that $f \in L^p(\mathbb{R}^d)$ for some $p \leq \frac{2d}{k}$. Then $f \equiv 0$.
- A natural question is whether the exponent $\frac{2d}{k}$ is **sharp**, and what does it have to with **restriction theory**? If $k = d - 1$ and S^{d-1} is the unit sphere, $\frac{2d}{d-1}$ is the sharp conjectured exponent for the dual of the restriction conjecture.

Proof of the Agranovsky-Narayanan theorem

- Let $\chi \in C_0^\infty$, supported on the unit ball,

$$\int \chi(x)dx = 1,$$

$$\chi_\epsilon(x) = \epsilon^{-d}\chi(x/\epsilon).$$

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- By Plancherel,

$$||u_\epsilon||_2 = \left(\int |f(x)|^2 |\widehat{\chi}(\epsilon x)|^2 dx\right)^{\frac{1}{2}} \lesssim ||f||_p \cdot \epsilon^{-\frac{d}{p'}}.$$

Proof of the Agranovsky-Narayanan theorem (continued)

- Let ψ be a smooth cut-off function. We have

$$| < u_\epsilon, \psi > |^2 \leq ||u_\epsilon||_2^2 \cdot \int_{S^\epsilon} |\psi(\xi)|^2 d\xi,$$

where S^ϵ is the ϵ-neighborhood of S.

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- The same argument works for any set of packing dimension k (not necessarily an integer).

Sharpness (or lack of it)

- If $S = S^{d-1}$, it is not difficult to see that the exponent $\frac{2d}{k} = \frac{2d}{d-1}$ is best possible since

$$\widehat{\sigma}_S(\xi) = J_{\frac{d-2}{2}}(|\xi|)|\xi|^{-\frac{d-2}{2}} \in L^p(\mathbb{R}^d) \text{ iff } p > \frac{2d}{d-1},$$

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where σ is the surface measure on S.

- On the other hand, if

$$S = \left\{(t, t^2, \dots, t^d) : t \in [0,1]\right\}, \ d \geq 3,$$

it is known that

$$\widehat{\sigma}_S \in L^p(\mathbb{R}^d) \text{ iff } p > \frac{d^2+d+2}{2} > \frac{2d}{k} = 2d.$$

A geometric approach to spectral synthesis

- Let $\widehat{f}$ be supported in S and let us cover S by a collection of **finitely overlapping** rectangles

$$\{R_{j,\delta}\}_{j=1}^{N(\delta)}, \ |R_{j,\delta}| \to 0 \text{ as } \delta \to 0.$$

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- Let $\mu_{j,\delta}$ denote a smooth partition of unity subordinate to $\{R_{j,\delta}\}_{j=1}^{N(\delta)}$. Since $\widehat{f}$ is supported in S, it is sufficient to consider

$$\widehat{f}(\xi) \cdot \sum_{j=1}^{N(\delta)} \mu_{j,\delta}(\xi), \text{ i.e.}$$

A geometric approach to spectral synthesis (continued)

- $$||f||_\infty \approx \left\| f * \sum_{j=1}^{N(\delta)} \widehat{\mu}_{j,\delta} \right\|_\infty \leq ||f||_p \cdot \left\| \sum_{j=1}^{N(\delta)} \widehat{\mu}_{j,\delta} \right\|_{p'}.$$

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- By Plancherel,

$$\left|\left| \sum_{j=1}^{N(\delta)} \widehat{\mu}_{j,\delta} \right|\right|_2 \approx \left(\sum_{j=1}^{N(\delta)} |R_{j,\delta}| \right)^{\frac{1}{2}} \equiv |S^\delta|^{\frac{1}{2}}.$$

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- Note that S^δ **is not necessarily** the δ-neighborhood of S.

A geometric approach to spectral synthesis (continued)

- On the other hand, since $R_{j,\delta}$'s are rectangles,

$$\left\| \sum_{j=1}^{N(\delta)} \widehat{\mu}_{j,\delta} \right\|_1 \lesssim \sum_{j=1}^{N(\delta)} |R_{j,\delta}| \cdot |R^*_{j,\delta}| = N(\delta).$$

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- The idea is to find the largest p for which this quantity $\to 0$ as $\delta \to 0$.

A flat example

- Suppose that S is a compact piece of a hyperplane. cover it with a single $1 \times 1 \times \cdots \times 1 \times \delta$ rectangle.

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- We conclude that
$$|S^\delta|^{\frac{1}{p}} \cdot (N(\delta))^{1-\frac{2}{p}} \approx \delta^{\frac{1}{p}},$$
which goes to 0 for any $p < \infty$.

A fun example

- Let $S = S^{d-1}$. Cover S by tangent $\delta^{\frac{1}{2}} \times \delta^{\frac{1}{2}} \times \ldots \delta^{\frac{1}{2}} \times \delta$ finitely overlapping rectangles. It is not difficult to see that

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- It follows that the critical value for p is $\frac{2d}{d-1}$, which is consistent with Agranovsky-Narayanan's theorem.

An even more entertaining example

- Let $S = \{(t, t^2, \dots, t^d) : t \in [0,1]\}$. Cover S by $\delta^{\frac{1}{d}} \times \delta^{\frac{2}{d}} \times \cdots \times \delta$ tangent rectangles.

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- We conclude that

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$$p_{critical} = \frac{d^2+d+2}{2}.$$

Space curves

Theorem

(S. Guo, A. Iosevich, R. Zhang, and P. Zorich-Kranich (2023)) Let $d \geq 2$ be a positive integer and suppose that $1 \leq p < \frac{d^2+d+2}{2}$. If $f \in L^p(\mathbb{R}^d)$ and $\widehat{f}$ is supported on

$$\{(t, t^2, \ldots, t^d) : t \in (0,1)\},$$

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- Note that the Agranovsky-Narayanan theorem yields the same conclusion for $p < 2d$ in this case.
- We also note that $\frac{d^2+d+2}{2}$ is the optimal extension exponent (more on that in a moment).

Connections with the restriction conjecture

- On the very first page of these notes, we discussed the restriction conjecture, which says that if S^{d-1} is the unit sphere, then

$$\left(\int_{S^{d-1}} \left|\widehat{f}(\xi)\right|^r d\sigma_S(\xi)\right)^{\frac{1}{r}} \leq C_{p,r}\left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{\frac{1}{p}}$$

whenever

$$p < \frac{2d}{d+1},\ r \leq \frac{d-1}{d+1}p',$$

where p' is the conjugate exponent to p.

Connections with the restriction conjecture

- On the very first page of these notes, we discussed the restriction conjecture, which says that if S^{d-1} is the unit sphere, then

$$\left(\int_{S^{d-1}} |\widehat{f}(\xi)|^r d\sigma_S(\xi)\right)^{\frac{1}{r}} \leq C_{p,r} \left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{\frac{1}{p}}$$

whenever

$$p < \frac{2d}{d+1}, \; r \leq \frac{d-1}{d+1} p',$$

where p' is the conjugate exponent to p.

- It is often convenient to state the dual of this inequality, the extension conjecture.

The extension conjecture

- The dual of the restriction conjecture above says that

$$||\widehat{f\sigma}||_{L^q(\mathbb{R}^d)} \leq C_{p,q}||f||_{L^p(S^{d-1})},$$

whenever

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- In general, if S is compact, equipped with Borel measure σ_S, we say that a (p,q)-extension estimate holds for S if

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- We call the inf of q's for which this estimate holds the critical extension exponent of S.

Extension versus spectral synthesis

- Based on examples we have so far, it seems reasonable to conjecture that if $\widehat{f}$ is supported in S, and $f \in L^p(\mathbb{R}^d)$ for p smaller than the critical extension exponent of S, then $f \equiv 0$.

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- I believe that it is possible to construct such a surface so that the critical extension exponent is $>> \frac{2d}{d-1}$.


Eh...
thanks
a lot!