ICMU
INTERNATIONAL CENTRE FOR MATHEMATICS IN UKRAINE

ICMU Summer School

PROBABILITY, GEOMETRY, AND MACHINE LEARNING

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Abstracts of minicourses

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Isoperimetric inequality, concentration of measure and log-concave densitiesy

The Euclidean isoperimetric inequality - that is, among bodies of given volume, the Euclidean balls minimize the surface area - was known already to the Ancient Greeks, but the first rigorous proof was only provided in the middle of the 19th century based on Steiner's symmetrization method. The main goal of the lecture series is to discuss connections between isoperimetric type inequalities and the concentration of measure phenomenon, which says that for certain probability spaces, the value of a Lipschitz function concentrates around its median with high probability.

The first talk discusses the proof of the Euclidean isoperimetric inequality via Steiner symmetrization. In addition, we consider a generalization, the so-called Brunn-Minkowski inequality for the linear combination of sets, and its functional analogue, the Prekopa-Leindler inequality.

The second talk focuses on the isoperimetric inequality on the sphere based on a suitable symmetrization, and in turn, using this in the Euclidean space, to prove the isoperimetric inequality for the Gaussian probability measure - note that half spaces are extremal in the latter case! We will see how both inequalities lead to the fundamental examples of the concentration of measure phenomenon.

The last talk discusses various inequalities, like the log-Sobolev inequality, Poincare inequality, Cheeger's inequality, to understand the concentration of measure phenomenon for more general log-concave densities.

Augusto Gerolin
University of Ottawa

Optimal transport and machine learning

In the last 30 years, the theory of optimal transport (OT) has emerged as a fertile field of inquiry and a diverse tool for exploring applications within and beyond mathematics, in such diverse fields as economics, meteorology, geometry, statistics, fluid mechanics and engineering. More recently, it has become one of the most important emerging topics in machine learning research.

Several fundamental challenges in Optimal Transport and applications are still open from all analytical, computational and statistical viewpoint. Motivated by these challenges, this mini-course aims to give a general introduction on mathematical, computational and statistical aspects of the OT theory as well as presenting open problems.

This course has minimal prerequisites, such as undergraduate-level probability, analysis, and linear algebra, and absolutely no contraindications. Everyone is warmly encouraged to attend.

Piotr Nayar
University of Warsaw

Positive pluses and negative pluses

The discrete cube, also called the hypercube or the Hamming cube, it the set $\Omega = \{-1,1\}^n$. One can equip Ω with probability space structure by considering the uniform probability distribution, in which case the coordinates of a uniform random vector in Ω are independent symmetric random signs. On the other hand, Ω has a natural graph structure with vertices $x, y \in \Omega$ being neighbors if and only if x and y differ on exactly one coordinate. Moreover, Ω admits a group structure being a direct product of n copies of $\mathbb{Z}_2$. These features make the mathematics of the hypercube very reach, leading to many interesting probabilistic and combinatorial questions, as well as allowing various proof techniques, including Fourier analytic methods. From the applied point of view, functions $f : \Omega \to \{-1,1\}$ can model voting procedures (with n voters and two candidates) in social choice theory and algorithms (with n bit input and one bit output) in computer science.

In the first part of this lecture series we develop the Walsh-Fourier analysis on Ω and use it to derive a Poincaré-type inequality leading to a simple proof of the $\ell_1 - \ell_2$ Khintchine inequality of Szarek from 1976, which solved a long-standing open problem due to Littlewood. We also give an application of the Fourier analytic method in the context of social choice theory, proving a version of Arrow's theorem on the existence of the so-called Condorcet winner in three-candidate elections. In the second part we shall discuss the classical Khintchine inequality in greater detail, focusing on the problem of finding best constants in various cases. We also relate Khinchine-type inequalities to the geometric problem of determining extremal (in terms of volume) sections and projections of unit balls in ℓ_p^n spaces. In the third part we discuss the isoperimetric inequality on the hypercube (Harper's theorem) and present a proof due to Hao Huang of the celebrated sensitivity conjecture. In the last part we present various theorems and problems related to geometric combinatorics, where random signs play a crucial role.

This includes Bang's solution to the Tarski's plank problem and the discussion of the so-called Komlós conjecture on balancing sums of unit vectors.

Mark Rudelson

University of Michigan

Limit distribution of the eigenvalues of a symmetric random matrix

Consider a family of symmetric random matrices whose entries above the main diagonal are independent and identically distributed. When the sizes of these matrices tend to infinity, the distribution of properly normalized eigenvalues converges to a probability measure having a semicircular density. This is the celebrated Wigner Semicircle Law which can be viewed as a non-commutative version of the Central Limit Theorem. In these lectures, we will prove this law while developing the necessary machinery along the way.

We will start with the introduction of the Stieltjes transform. This is the main technical tool playing the role that the Fourier transform plays in the proof of the Central Limit Theorem. We will discuss its basic properties and informally establish the self-consistent equation for this transform which will in turn imply the Semicircle Law. A formal derivation of this equation requires another major tool, namely the Hanson-Wright inequality. This inequality from high dimensional probability establishes concentration of a random quadratic form around its expectation. After proving the Hanson-Wright inequality, we will rigorously derive the self-consistent equation thus completing the proof of the Semicircular Law.

If time allows, we will discuss a similar analytic approach to the Circular Law which is another central result in random matrix theory.