ICMU
INTERNATIONAL CENTRE FOR MATHEMATICS IN UKRAINE

ICMU Summer School

OPERATOR ALGEBRAS, QUANTUM GROUPS AND QUANTUM INFORMATION THEORY

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Abstracts of mini-courses

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Tomita-Takesaki theory and applications to quantum groups

Let M be a von Neumann algebra with φ, a faithful normal state (or weight). If φ is tracial, then the map $S_\varphi\colon \Lambda_\varphi(x) \mapsto \Lambda_\varphi(x^*)$ (densely defined on the GNS Hilbert space) extends to an antiunitary. When φ is non-tracial, one can still consider such a map S_φ, but it will no longer be bounded; let $S_\varphi = J_\varphi \nabla_\varphi^{1/2}$ be its polar decomposition. The celebrated Tomita-Takesaki theory gives us some tools to handle this situation. It turns out that $J_\varphi M J_\varphi$ is equal to M', the commutant of M, and $\sigma_t^\varphi(m) = \nabla_\varphi^{it} m \nabla_\varphi^{-it}$ defines a $*$-automorphism of M (called modular automorphism). These two claims are the main results of the Tomita-Takesaki theory. Its introduction lead to main advancements in theory of (type III) von Neumann algebras. One example of how modular automorphisms can be used handle non-traciality of φ, is the equality $\varphi(xy) = \varphi(y\sigma_{-i}^\varphi(x))$. In fact, φ is tracial if, and only if automorphisms σ_t^φ are trivial.

A motivational example of this situation is given by $L^\infty(\mathbb{G})$, the von Neumann algebra of bounded functions on a compact quantum group $\mathbb{G}$, together with Haar integral $h \in L^\infty(\mathbb{G})_*$, which is a faithful normal state. In many important situations (e.g. $\mathbb{G} = SU_q(2)$ or $\mathbb{G} = O_F^+$) Haar integral h is non-tracial and σ_t^h are interesting automorphisms.

During the course I will present main results of Tomita-Takesaki theory, with particular focus on examples and applications in compact quantum group theory.

Slavik Rabanovich

National Academy of Sciences of Ukraine, Institute of Mathematics

Completely positive maps and quantum channels with special type of Kraus operators

We are going to talk about linear maps between matrix spaces and concentrate our attention on completely positive maps (CP maps). We shall consider some methods of constructing such maps in particular using Kraus decomposition and Choi matrix. After that we will discuss quantum channel as a CP map with restrictions. Using linear combinations of projections as examples of special quantum channels, we shall discuss some mathematical problems of quantum information theory.

Adam Skalski

Institute of Mathematics of the Polish Academy of Sciences (IMPAN)

Introduction to compact quantum groups

We will begin the lectures by recalling key basic facts regarding compact groups and their representations. Then we shall pass to noncommutative mathematics, explaining (some of) the motivations behind the introduction of compact quantum groups. We will introduce the definition due to Woronowicz, establish the existence of Haar state of a compact quantum group and present the Woronowicz-Peter-Weyl representation theory, together with some of its consequences. The lectures will end by outlining several examples of compact quantum groups and discussing some of the current research in the area.

Lyudmyla Turowska

Chalmers University of Technology and University of Gothenburg

Operator algebras and non-local games

There are several mathematical models to describe the conditional probability densities/correlations that can occur when two labs in entangled state conduct quantum measurements. It has been fundamental research done to study whether these models give rise to the same sets of correlations. The celebrated Bell theorem demonstrates that the set of classical correlations are strictly smaller than the quantum ones, while the Tsirelsson problems are related to differences between "physically realizable" bipartite probability distributions. Operator systems, operator algebras and their tensor products have been an important tool to study such distributions. One of the Tsirelson problems is related to finite approximability in operator algebras and is equivalent to the Connes' embedding problem in von Neumann algebras, the solution of which was recently announced.

Many results on correlations have come from the study of non-local games and their winning strategies. They witness the differences between classes of correlations and provide ways of constructing new interesting classes of operator algebras.

In these lectures I will give an introduction of non-local games. After going over the basic theory of operator systems I will highlight the role C*-algebras and operator systems play in mathematical understanding of quantum correlations and perfect strategies of non-local games. Synchronous games as for example graph homomorphism/isomorphism games are of particular interests as their perfect strategies can be described through traces of affiliated C*-algebras. I will discuss differences between classes of quantum correlations.